ON THE IWASAWA λ-INVARIANTS OF CERTAIN REAL ABELIAN FIELDS II

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§1 Introduction

For a number field k and a prime number p, denote respectively by $\lambda_p(k)$ and $\mu_p(k)$ the Iwasawa λ-invariant and the μ-invariant associated to the ideal class group of the cyclotomic $\mathbb{Z}_p$-extension $k_\infty = \cup k_n$ over k with k_n its n-th layer($n \geq 0$). It is conjectured that $\lambda_p(k) = \mu_p(k) = 0$ for any totally real number field k and any p([21,page 316], [13]), which is often called Greenberg's conjecture. As for μ-invariants, we always have $\mu_p(k) = 0$ when k is abelian over $\mathbb{Q}$([10]).

We continue our previous investigations[17], [18] and establish an effective and simple method to verify the conjecture for each real abelian field. For our purpose, a point is how to deal with (global) units of k_n for each n or those of k_∞.

Let p be a fixed odd prime number, χ a $\overline{\mathbb{Q}}_p$-valued nontrivial even primitive Dirichlet character and k the real abelian field associated to χ. We work under the(rather technical) assumption that the exponent of $\Delta = \mathrm{Gal}(k/\mathbb{Q})$ divides $p-1$ for the convenience of computation. Denote by $A_\infty = A_\infty(k)$ the projective limit of the Sylow p-subgroup of the ideal class group of each layer k_n of k_∞/k w.r.t. the relative norms. As is well known, A_∞ is finitely generated and torsion over the power series ring $\Lambda = \mathbb{Z}_p[[T]]$ with $1+T$ acting as a fixed topological generator of $\mathrm{Gal}(k_\infty/k)$ ([21,Theorem 5]). Greenberg's conjecture for (p, χ) is

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX stated that the λ-invariant $\lambda_\chi = \lambda(A_\infty(\chi))$ of the χ-component $A_\infty(\chi)$ vanishes. We have an upper bound for λ_χ and the characteristic polynomial $\mathrm{char}\, A_\infty(\chi)$ of $A_\infty(\chi)$. Let $P_\chi(T)(\in \mathbb{Z}_p[T])$ be the distinguished polynomial associated to the p-adic L-function $L_p(s,\chi)$ (see (1),(2) of §2), and put $\lambda_\chi^* = \deg P_\chi(T)$. Then, as a consequence of the Iwasawa main conjecture(proved by [24]), we have $\mathrm{char}\, A_\infty(\chi) \mid P_\chi(T)$ and hence $\lambda_\chi \leq \lambda_\chi^*$(see (3) of §2).

In [17], we have given an effective and simple criterion for $\lambda_\chi = 0$ in the simplest case where $\lambda_\chi^* = 1$ (and p satisfies some additional conditions). When $k = \mathbb{Q}(\cos 2\pi/p)$, we have given in [18] a criterion of style similar to [17] without the assumption $\lambda_\chi^* = 1$ but under a "rationality assumption" on the zeros of $P_\chi(T)$. Inspired by the results/ideas of these works, we have succeeded in obtaining a nice criterion for $\lambda_\chi = 0$ in the general case.

Let $P(T)$ be any irreducible factor of $P_\chi(T)$. Our main theorem asserts that $P(T) \nmid \mathrm{char}\, A_\infty(\chi)$ if and only if a condition $(H_{P,n})$ given for each $n \geq 0$ holds for some n. The condition $(H_{P,n})$ is written, in a very simple form, in terms of an explicitly given cyclotomic unit c_n of k_n and a polynomial $X_{P,n}(T)$. This polynomial $X_{P,n}$ is defined for $P(T)$ and $n(\geq 0)$ in a very simple way, but it is so powerful that it enables us to "handle" the group of global units of k_n. When $P = T - \alpha$ is of degree one, $X_{P,n}$ equals to the polynomial $\frac{\omega_n(T)-\omega_n(\alpha)}{T-\alpha}$ (times a unit of $\mathbb{Z}_p$), which already appeared and played a role in [17], [18]. Here, $\omega_n(T) = (1+T)^{p^n} - 1$.

From our theorem(and the above mentioned upper bound for $\mathrm{char}\, A_\infty(\chi)$), we obtain a simple criterion for $\lambda_\chi = 0$(Corollaries 1,2). In a sense, our criterion is a natural generalization of the classical one for the Vandiver conjecture ($p \nmid h(\mathbb{Q}(\cos 2\pi/p))$), which is also given in terms of cyclotomic units of $\mathbb{Q}(\mu_p)$(cf. [29,Corollary 8.19]). Since the cyclotomic unit c_n and the polynomial $P_\chi(T)$ (and, hence, $P(T)$ and $X_{P,n}(T)$) are quite easy to handle in a computational sense, our criterion is very suitable for practical computer calculation. Using

Corollary 2 of Theorem, we have seen by some computation that $\lambda_3(k) = 0$ for all real quadratic fields $k = \mathbb{Q}(\sqrt{m})$ with m square free and $1 < m < 10^4$.

For real quadratic fields(and $p = 3$), several authors([2], [7], [8], [9], [13], [25], [26], [27]) have given, for each $n \geq 0$, some sufficient conditions for $\lambda_p(k) = 0$ in terms of mainly the fundamental units of k_n, (which can be dealt with only for small n). Some of them use also the polynomial $P_\chi(T)$. By using these criterions, it is already known that $\lambda_3(k) = 0$ for many real quadratic fields in the above range. Remaining ones were $\mathbb{Q}(\sqrt{254})$, $\mathbb{Q}(\sqrt{473})$, $\mathbb{Q}(\sqrt{9814})$, etc. These are "tough" in the sense that one can not prove $\lambda_3(k) = 0$ using data of k_n for small n by the methods known at present or by ours. But, the method established in [17], [18] and this paper is efficient also for such tough ones.

During the preparation of this paper, J.S. Kraft and R. Schoof[22] and M. Kurihara[23] obtained some methods to verify $\lambda_\chi = 0$ for certain real abelian fields, which are efficient also for tough ones. [22] deals with real quadratic fields k with $\chi(p) \neq 1$, χ being the associated Dirichlet character. [22] also studies the Λ-structure of $A_\infty(\chi)$ not only the invariant λ_χ. [23] works mainly under the assumption that $A_\infty(\chi)$ is cyclic over $\mathbb{Z}_p$. By some computation, they add new examples with $\lambda_\chi = 0$. The methods of ours, [22] and [23] are different and obtained from different viewpoints. But, in practical computational application, all these methods require some calculation of certain cyclotomic units modulo several prime ideals. A characteristic of ours compared with [22], [23] and other related works is that we have established a new way to apply p-adic L-functions, introducing the polynomial $X_{P,n}(T)$. We believe that this polynomial $X_{P,n}$, though very simple, plays some role in some further investigation on cyclotomic fields.

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§2 Main Theorem

Let p be a fixed odd prime number, χ a fixed $\overline{\mathbb{Q}}_p$-valued nontrivial even primitive Dirichlet character and k the real abelian field associated to χ. Here, $\overline{\mathbb{Q}}_p$ denotes a fixed algebraic closure of $\mathbb{Q}_p$. Let k_∞/k be the cyclotomic $\mathbb{Z}_p$-extension with its n-th layer k_n ($n \geq 0$). Let A_n be the Sylow p-subgroup of the ideal class group of k_n, and put $A_\infty = \varprojlim A_n$, the projective limit being taken w.r.t. the relative norms. Put $\Delta = \mathrm{Gal}(k/\mathbb{Q})$, $\Gamma = \mathrm{Gal}(k_\infty/k)$ and $G_\infty = \mathrm{Gal}(k_\infty/\mathbb{Q})$. We regard Δ as a subgroup of G_∞ in an obvious way;$G_\infty = \Delta \times \Gamma$. We assume the following rather technical condition for the convenience of computation.

(C1) The exponent of Δ divides $p-1$.

Let

$$e_\chi = \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \chi(\sigma)\sigma^{-1}$$

be the idempotent of $\overline{\mathbb{Q}}_p[\Delta]$ corresponding to χ. By (C1), e_χ is an element of $\mathbb{Z}_p[\Delta]$. For a $\mathbb{Z}_p[\Delta]$-module M, denote by $M(\chi)$ the χ-component $e_\chi M$(or M^{e_χ}) of M. Let f be the conductor of χ and q the least common multiple of f and p. Identifying Γ with $\mathrm{Gal}(k(\mu_{p^\infty})/k(\mu_p))$ in a natural way, we choose and fix the topological generator γ of Γ such that $\zeta^\gamma = \zeta^{1+q}$ for all $\zeta \in \mu_{p^\infty}$. We identify, as usual, the completed group ring $\mathbb{Z}_p[[\Gamma]]$ with the power series ring $\Lambda = \mathbb{Z}_p[[T]]$ by $\gamma = 1+T$. Thus, for a $\mathbb{Z}_p[[G_\infty]]$-module M (e.g. $M = A_\infty$), $M(\chi)$ is regarded as a module over $\Lambda = \mathbb{Z}_p[[T]]$. For a finitely generated torsion Λ-module M, denote its characteristic polynomial by $\mathrm{char}\, M$. As is well known, $A_\infty(\chi)$ is finitely generated and torsion over Λ ([21,Theorem 5]). We denote respectively by λ_χ and μ_χ the λ-invariant and the μ-invariant of the torsion Λ-module $A_\infty(\chi)$ (or, equivalently, of the power series $\mathrm{char}\, A_\infty(\chi)$). We know that $\mu_\chi = 0$([10]). Hence, $\mathrm{char}\, A_\infty(\chi) = 1$ if and only if $\lambda_\chi = 0$. Greenberg's conjecture for (p, χ) is stated as follows:

Conjecture(p, χ) $\qquad \mathrm{char}\, A_\infty(\chi) = 1,$ namely, $\lambda_\chi = 0.$

By [20](and (C1)), there exists uniquely a power series $g_\chi(T)$ in $\mathbb{Z}_p[[T]]$ related to the p-adic L-function $L_p(s, \chi)$ by

$$g_\chi((1+q)^{1-s} - 1) = L_p(s, \chi). \tag{1}$$

By the p-adic Weierstrass preparation theorem, we can write uniquely

$$g_\chi(T) = p^\mu P_\chi(T) u_\chi(T) \tag{2}$$

for an integer $\mu = \mu_\chi^* (\geq 0)$, a distinguished polynomial $P_\chi(T)$ and a unit $u_\chi(T)$ of Λ. By [10], $\mu_\chi^* = 0$. Put $\lambda_\chi^* = \deg P_\chi(T)$. By the Iwasawa main conjecture proved in [24](see Fact3 in §3-2), we have an upper bound for λ_χ and $\mathrm{char}\, A_\infty(\chi)$:

$$\lambda_\chi \leq \lambda_\chi^*, \quad \mathrm{char}\, A_\infty(\chi) \mid P_\chi(T). \tag{3}$$

Let χ^* be the odd primitive Dirichlet character induced from $\omega\chi^{-1}$, here ω is the Teichmüller character $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p$. We divide the character χ into the following three classes:

(A) $\qquad \chi(p) \neq 1$ and $\chi^*(p) \neq 1,$

(B) $\qquad \chi^*(p) = 1,$

(C) $\qquad \chi(p) = 1.$

As is easily seen, the conditions (B) and (C) can not simultaneously hold. The reasons for dividing χ as above are (1) that the behavior of $L_p(s, \chi)$ at $s = 0$ are different between the cases (A), (C) and the case (B), and (2) that the structures of local units of k_n at the primes over p are slightly different between any two of the three classes(cf. §3-1).

Assume that χ satisfies (B). Then, $L_p(s,\chi)$ has a zero(trivial zero) at $s=0$, and it is a single zero ([4,Proposition 2]). Hence, by (1), we have

$$(4_B) \qquad T-q \parallel P_\chi(T).$$

Further, we see in §3-2(Remark 3) that

$$(5_B) \qquad T-q \nmid \operatorname{char} A_\infty(\chi).$$

Thus, we have

$$(5'_B) \qquad \lambda_\chi^* \geq 1 \text{ and } \lambda_\chi \leq \lambda_\chi^* - 1.$$

By (3)(resp. $(5'_B)$), we have $\lambda_\chi = 0$ if $\lambda_\chi^* = 0$(resp. $\lambda_\chi^* = 1$) in the case (A) or (C)(resp. (B)). So, we further assume the following to exclude this "trivial" case.

(C2) $\qquad \lambda_\chi^* \geq 1$(resp. $\lambda_\chi^* \geq 2$) in the case (A) or (C)(resp. (B)).

We decompose (by using (4_B) in the case (B))

$$(6) \qquad P_\chi(T) = \begin{cases} P_1(T)^{e_1}\cdots P_r(T)^{e_r} & \text{in the case (A) or (C)} \\ (T-q)P_1(T)^{e_1}\cdots P_r(T)^{e_r} & \text{in the case (B)} \end{cases}$$

for some $r \geq 1$, some irreducible distinguished polynomials $P_i(T)$ in $\mathbb{Z}_p[T](1 \leq i \leq r)$ with $P_i \neq P_j (i \neq j)$ and, in the case (B), $P_i \neq T-q$, and some natural numbers e_i. Put $\omega_n = \omega_n(T) = (1+T)^{p^n} - 1$ $(n \geq 0)$ and $\nu_n = \nu_n(T) = \omega_n/T(n \geq 1)$. By the Leopoldt conjecture for (k_n, p) proved in [1], $P_i(T)$ and $\omega_n(T)$ are relatively prime(see (17)). So, the abelian groups $\Lambda/(P_i,\omega_n)$ and $\Lambda/(P_i,\nu_n)$ are finite. We denote by $p^{a_{i,n}}$ the exponent of $\Lambda/(P_i,\omega_n)$(resp.

$\Lambda/(P_i, \nu_n))$ in the case (A) or (B)(resp. (C)). Then, we can take a polynomial $X_{P_i,n}(T) = X_{i,n}(T)$ in $\mathbb{Z}_p[T]$ satisfying

$$(7_n) \qquad X_{i,n}P_i \equiv p^{a_{i,n}} \bmod \begin{cases} \omega_n & \text{in the case (A) or (B)}(n \geq 0) \\ \nu_n & \text{in the case (C)}(n \geq 1). \end{cases}$$

This polynomial $X_{i,n}$ is uniquely determined modulo ω_n(resp. ν_n) as $P_i \nmid \omega_n$, and plays a very important role in this paper. Define a polynomial $Y_{i,n}(T)$ in $\mathbb{Z}[T]$ by

$$(8) \qquad Y_{i,n}(T) \equiv X_{i,n}(T) \bmod p^{a_{i,n}}.$$

Let $\mathbf{e}_{\chi,i,n}$ be an element of $\mathbb{Z}[\Delta]$ such that

$$(9) \qquad \mathbf{e}_{\chi,i,n} \equiv e_\chi \bmod p^{a_{i,n}}$$

and the sum of its coefficients is zero. Put

$$(10) \qquad c_n = N_{\mathbb{Q}(\mu_{f_n})/k_n}(1-\zeta_{f_n})^{t-1}, \quad c_{i,n} = c_n^{\mathbf{e}_{\chi,i,n}} (\in k_n^\times).$$

Here, f_n is the conductor of k_n, ζ_{f_n} a fixed primitive f_n-th root of unity and t the cardinality of the residue class field of a prime ideal of k over p. Since the sum of coefficients of $\mathbf{e}_{\chi,i,n}$ is zero, $c_{i,n}$ is a unit (cyclotomic unit) of k_n. By the identification $\gamma = 1+T$, the polynomial $Y_{i,n}(T) (\in \mathbb{Z}[T])$ can act on any element of the multiplicative group $k_n^\times$. For each $i (1 \leq i \leq r)$ and $n \geq 0 (n \geq 1$ in the case (C)), consider the following condition:

$$(H_{P_i,n}) = (H_{i,n}) \qquad c_{i,n}^{Y_{i,n}(T)} \notin k_n^{\times p^{a_{i,n}}}$$

We see at the end of §4

Lemma 1. *The condition $(H_{i,n})$ implies $(H_{i,n+1})$.*

Now, our main theorem is stated as follows.

Theorem. *Assume that (p,χ) satisfies $(C1)$ and $(C2)$. Then, for each $i(1 \leq i \leq r)$, we have $P_i(T) \nmid \operatorname{char} A_\infty(\chi)$ if and only if the condition $(H_{i,n})$ holds for some $n \geq 0 (n \geq 1$ in the case $(C))$.*

By (3)(and (4_B), (5_B) in the case (B)), we obtain from the above

Corollary 1. *Under the above setting, we have $\lambda_\chi = 0$ if and only if for any $i(1 \leq i \leq r)$, the condition $(H_{i,n})$ holds for some $n \geq 0$ $(n \geq 1$ in the case $(C))$.*

By the Chebotarev density theorem, we finally obtain

Corollary 2. *Under the above setting, we have $\lambda_\chi = 0$ if and only if for any $i(1 \leq i \leq r)$, there exist an integer $n \geq 0$ $(n \geq 1$ in the case $(C))$ and a prime ideal $\mathfrak{l}$ of k_n of degree one for which the condition*

$$c_{i,n}^{Y_{i,n}} \bmod \mathfrak{l} \notin (\mathbb{Z}/l\mathbb{Z})^{\times p^{a_{i,n}}}$$

holds. Here, $l = \mathfrak{l} \cap \mathbb{Q}$.

§3 Preliminaries for proving Theorem

§3-1 Local units modulo cyclotomic units

We assume that the couple (p,χ) satisfies (C1), and use the same notations as in §2.Let $\mathfrak{p}$ be a prime ideal of k over p and $\mathfrak{p}_n$ the unique prime ideal of k_n over $\mathfrak{p} = \mathfrak{p}_0$. Denote by $k_{\mathfrak{p}_n} (\subset \overline{\mathbb{Q}}_p)$ the completion of k_n at $\mathfrak{p}_n$, and by $\mathcal{U}_{\mathfrak{p}_n}$ the group of principal units of $k_{\mathfrak{p}_n}$. Put $\mathcal{V}_{\mathfrak{p}_n} = \bigcap_{m \geq n} N_{m/n} \mathcal{U}_{\mathfrak{p}_m}$, $N_{m/n}$ denoting the norm map. We put

$$\mathcal{U}_n = \prod_{\mathfrak{p}|p} \mathcal{U}_{\mathfrak{p}_n} \quad \text{and} \quad \mathcal{V}_n = \prod_{\mathfrak{p}|p} \mathcal{V}_{\mathfrak{p}_n},$$

where $\mathfrak{p}$ runs over all prime ideals of k over p.

Let E'_n be the group of units ε of k_n satisfying $\varepsilon \equiv 1 \bmod \mathfrak{p}_n$ for all $\mathfrak{p} \mid p$. Denote by C_n the subgroup of $k_n^\times$ generated by all the units

$$N_{\mathbb{Q}(\mu_{f_n})/k_n}(1 - \zeta_{f_n})^u, \quad u \in \mathbb{Z}[\mathrm{Gal}(k_n/\mathbb{Q})]^0.$$

Here, for a group ring X, X^0 denotes the augmentation ideal. The unit $c_{i,n}$ defined by (10) is an element of C_n. (The group C_n is smaller than the group $\tilde{C}_n$ of cyclotomic units of k_n in the sense of [15] and [11,§2-3]. A relation between C_n and $\tilde{C}_n$ is given in [12,p.9].) Denote respectively by $\mathcal{E}_n$ and $\mathcal{C}_n$ the closures of the images of E'_n and $C_n \cap E'_n$ under the diagonal embedding $E'_n \to \mathcal{U}_n$. Put

$$\mathcal{U} = \varprojlim \mathcal{U}_n, \quad \mathcal{E} = \varprojlim \mathcal{E}_n, \quad \mathcal{C} = \varprojlim \mathcal{C}_n,$$

the projective limits being taken w.r.t. the relative norms. By the definition of $\mathcal{V}_n$, we have

$$\mathcal{U} = \varprojlim \mathcal{V}_n.$$

On the groups $\mathcal{U}_n$, $\mathcal{U}$ etc. defined above, Δ and Γ act in a natural way. Thus, $\mathcal{U}_n(\chi)$, $\mathcal{U}(\chi)$ etc. can be viewed as Λ-modules.

As is easily seen, we have

$$\mathcal{U}_n(\chi) = \mathcal{V}_n(\chi)(n \geq 0) \quad \text{in the case (A) or (B)}, \tag{11}$$

$$N_{n/0}\mathcal{V}_n(\chi) = \mathcal{V}_0(\chi) = \{1\}(n \geq 1) \quad \text{in the case (C)}, \tag{12}$$

$$\mathbb{T}_n := \mathrm{Tor}_{\mathbb{Z}_p}\mathcal{U}_n(\chi) = \mu_{p^{n+1}} \quad \text{in the case (B)}. \tag{13}$$

Here, $\mathrm{Tor}_{\mathbb{Z}_p}$ denotes the torsion submodule as a $\mathbb{Z}_p$-module. For (11)-(13), see [12,Propositions 1,2]. In the case (B), we put

$$\mathbb{T} := \varprojlim \mathbb{T}_n (\subset \mathcal{U}(\chi)).$$

We easily see from the definition of C_n that $\mathcal{C}_n(\chi)$ is generated over Λ by

$$c_{\chi,n} = N_{\mathbb{Q}(\mu_{f_n})/k_n}(1 - \zeta_{f_n})^{(t-1)e_\chi} \tag{14}$$

and that

$$\mathcal{C}_n(\chi) \subset \mathcal{V}_n(\chi). \tag{15}$$

The following fact obtained in [19] and [12] plays an important role when we deal with the groups $\mathcal{U}_n(\chi)/\mathcal{E}_n(\chi)$ and $\mathcal{U}(\chi)/\mathcal{E}(\chi)$ in §4.

Fact 1([12,Theorems 1,2, Propositions 1,2]). *Assume that* (p,χ) *satisfies* $(C1)$. *In the case* (B), *put* $\tilde{P}_\chi(T) = P_\chi(T)/(T-q)$. *We have the following* Λ *-isomorphisms according as* χ *satisfies* (A),(B) *or* (C).

$$\text{(A)}\quad \begin{array}{ccc} \mathcal{U}(\chi)\simeq\Lambda & \mathcal{U}_n(\chi)\simeq\Lambda/(\omega_n) & \\ \cup \qquad \cup & \cup \qquad \cup & n\geq 0 \\ \mathcal{C}(\chi)\simeq(P_\chi), & \mathcal{C}_n(\chi)\simeq(P_\chi,\omega_n)/(\omega_n). & \end{array}$$

$$\text{(B)}\quad \begin{array}{ccc} \mathcal{U}(\chi)/\mathbb{T}\simeq\Lambda & \mathcal{U}_n(\chi)/\mathbb{T}_n\simeq\Lambda/(\omega_n) & \\ \cup \qquad \cup & \cup \qquad \cup & n\geq 0 \\ \mathcal{C}(\chi)\mathbb{T}/\mathbb{T}\simeq(\tilde{P}_\chi), & \mathcal{C}_n(\chi)\mathbb{T}_n/\mathbb{T}_n\simeq(\tilde{P}_\chi,\omega_n)/(\omega_n). & \end{array}$$

$$\text{(C)}\quad \begin{array}{ccc} \mathcal{U}(\chi)\simeq\Lambda & \mathcal{V}_n(\chi)\simeq\Lambda/(\nu_n) & \\ \cup \qquad \cup & \cup \qquad \cup & n\geq 1 \\ \mathcal{C}(\chi)\simeq(P_\chi), & \mathcal{C}_n(\chi)\simeq(P_\chi,\nu_n)/(\nu_n). & \end{array}$$

We also need the following lemma on E'_n and $\mathcal{E}_n$, which follows from the Leopoldt conjecture for (k_n,p)(proved in [1]).

Lemma 2. (cf. [29,§5-5]) *The inclusion* $E'_n\to\mathcal{E}_n$ *induces an isomorphism*

$$E'_n/{E'_n}^{p^a}\simeq\mathcal{E}_n/\mathcal{E}_n^{p^a}$$

for any $a\geq 1$.

Remark 1. Since E'_n has no primitive p-th root of unity, so does $\mathcal{E}_n(\chi)$ by the above lemma.

§3-2 Inertia groups

We use the same notations as before. Let M/k_∞ be the maximal pro-p abelian extension unramified outside p and L/k_∞ the maximal unramified pro-p abelian extension. Denote by M_n(resp. L_n) the maximal abelian extension of k_n contained in M(resp. L). Put

$$\mathcal{G}=\mathrm{Gal}(M/k_\infty),\quad I=\mathrm{Gal}(M/L)\text{ and }I_n=\mathrm{Gal}(M_n/L_n).$$

Then, we have $I = \varprojlim I_n$, the projective limit being taken w.r.t. the restriction map $I_{n+1} \to I_n$. The Galois groups defined above are considered as $\mathbb{Z}_p[[G_\infty]]$-modules in a natural way, hence their χ-components are regarded as Λ-modules. As is well known, $\mathcal{G}(\chi)$ and $I(\chi)$ are finitely generated and torsion over Λ([21,Theorems 5,16]). By class field theory, we have a canonical Λ-isomorphism

(16) $$A_\infty(\chi) \simeq \mathcal{G}(\chi)/I(\chi).$$

Thus, Greenberg's conjecture for (p, χ) (namely, $\mathrm{char}\, A_\infty(\chi) = 1$) is equivalent to

Conjecture'(p, χ) $$\mathrm{char}\, \mathcal{G}(\chi) = \mathrm{char}\, I(\chi).$$

As for the whole Galois group $\mathcal{G}(\chi)$, we have:

Fact 2 $\mathcal{G}(\chi)$ has no nontrivial finite Λ-submodule([21,Theorem 18]).

Fact 3 $$\mathrm{char}\, \mathcal{G}(\chi) = P_\chi(T).$$

Fact 3 is a version of the Iwasawa main conjecture proved in [24] (cf. [12,p.13]). The Leopoldt conjecture for (k_n, p) proved in [1] asserts that $\mathrm{Gal}(M_n/k_\infty)$ is finite(cf. [13, p.265-266]). Thus, from Fact 3, we must have

(17) $P_\chi(T)$ and ω_n are relatively prime for all $n \geq 0$.

As for the inertia groups I and I_n, the following lemma, a consequence of class field theory, is fundamental.

Lemma 3. *Assume that* (p,χ) *satisfies* $(C1)$. *We have the following* Λ-*isomorphisms:*

$$I(\chi)\simeq \mathcal{U}(\chi)/\mathcal{E}(\chi).$$
$$I_n(\chi)\simeq \mathcal{V}_n(\chi)\mathcal{E}_n(\chi)/\mathcal{E}_n(\chi)\simeq \begin{cases} \mathcal{U}_n(\chi)/\mathcal{E}_n(\chi) & \text{in the case (A) or (B)} \\ \mathcal{V}_n(\chi)/\mathcal{V}_n(\chi)\cap\mathcal{E}_n(\chi) & \text{in the case (C).} \end{cases}$$

Proof. Though this assertion seems to be more or less known, we give a proof as we could not find an appropriate reference. Since $I=\varprojlim I_n$, it suffices to show the second isomorphism. Further, by considering k_n as a base field(and by (11)), it suffices to deal with the case $n=0$ and show $I_0\simeq \mathcal{V}_0/\mathcal{V}_0\cap\mathcal{E}_0$ as $\mathbb{Z}_p[\mathrm{Gal}(k/\mathbb{Q})]$-modules.

Let K_0 be the Hilbert p-class field of k and $M_{0,m}$ the maximal abelian extension of K_0 contained in M_0 whose Galois group over K_0 is of exponent p^m. Clearly, $M_{0,m}\supseteq K_0k_m$. Denote by $L_{0,m}$ the maximal unramified abelian extension of K_0k_m contained in $M_{0,m}$. Then we have

$$I_0\simeq \varprojlim \mathrm{Gal}(M_{0,m}/L_{0,m}),$$

the projective limit being taken w.r.t. the restriction maps.

Let J_k be the idéle group of k. Put

$$\mathcal{U}'_{(m)}=\{(u_\mathfrak{p})\in\prod_{\mathfrak{p}|p}\mathcal{U}_\mathfrak{p}\mid \prod_{\mathfrak{p}|p}\left(\frac{u_\mathfrak{p},k_m/k}{\mathfrak{p}}\right)=1\}.$$

Here, $\mathcal{U}_\mathfrak{p}=\mathcal{U}_{\mathfrak{p}_0}$(in the notation of §3-1) and $\left(\frac{*,k_m/k}{\mathfrak{p}}\right)$ denotes the norm residue symbol. For a prime divisor $\mathfrak{q}$ of k with $\mathfrak{q}\nmid p$, we denote by $\mathcal{U}_\mathfrak{q}$ the group of local units(resp. the multiplicative group) of the completion $k_\mathfrak{q}$ of k at $\mathfrak{q}$ when q is finite(resp. infinite). By class field theory, we have a canonical isomorphism

$$\mathrm{Gal}(K_0k_m/k)\simeq (J_k/k^\times(\mathcal{U}'_{(m)}\times\prod_{\mathfrak{q}\nmid p}\mathcal{U}_\mathfrak{q}))(p).$$

Here, in the product $\prod_{\mathfrak{q}\nmid p}$, $\mathfrak{q}$ runs over all prime divisors of k with $\mathfrak{q} \nmid p$, and for a finite abelian group X, $X(p)$ denotes the Sylow p-subgroup. As is easily seen, the subgroup of $H_1 = k^\times(\mathcal{U}'_{(m)} \times \prod_{\mathfrak{q}\nmid p}\mathcal{U}_\mathfrak{q})$ corresponding to $M_{0,m}(\supseteq K_0k_m)$ by the reciprocity law map is

$$H_2 = k^\times(\prod_{\mathfrak{p}|p}\mathcal{U}_\mathfrak{p}^{p^m} \times \prod_{\mathfrak{q}\nmid p}\mathcal{U}_\mathfrak{q}).$$

Namely, we have $\mathrm{Gal}(M_{0,m}/K_0k_m) \simeq H_1/H_2$. On the other hand, the subgroup of H_1 corresponding to $L_{0,m}$ is

$$H_3 = \langle (k^\times(U_\mathfrak{p} \times \prod_{\mathfrak{q}\neq\mathfrak{p}}1)H_2) \cap H_1 \mid \mathfrak{p} \mid p\rangle.$$

Namely, we have $\mathrm{Gal}(L_{0,m}/K_0k_m) \simeq H_1/H_3$. Here, in the product $\prod_{\mathfrak{q}\neq\mathfrak{p}}$, $\mathfrak{q}$ runs over all prime divisors of k with $\mathfrak{q} \neq \mathfrak{p}$. We easily see that

$$H_3 = k^\times(\prod_{\mathfrak{p}|p}N_{m/0}\mathcal{U}_{\mathfrak{p}_m} \times \prod_{\mathfrak{q}\nmid p}\mathcal{U}_\mathfrak{q})$$

by using the product formula for the norm residue symbols. Thus, we have

$$\begin{aligned}\mathrm{Gal}(M_{0,m}/L_{0,m}) &\simeq H_3/H_2 = (\prod_{\mathfrak{p}|p}N_{m/0}U_{\mathfrak{p}_m} \times \prod_{\mathfrak{q}\nmid p}1)H_2/H_2 \\ &\simeq (\prod_{\mathfrak{p}|p}N_{m/0}\mathcal{U}_{\mathfrak{p}_m} \times \prod_{\mathfrak{q}\nmid p}1)/(\prod_{\mathfrak{p}|p}N_{m/0}\mathcal{U}_{\mathfrak{p}_m} \times \prod_{\mathfrak{q}\nmid p}1) \cap H_2.\end{aligned}$$

As is easily seen, the latter group is naturally isomorphic to

$$\prod_{\mathfrak{p}|p}N_{m/0}\mathcal{U}_{\mathfrak{p}_m}/(\prod_{\mathfrak{p}|p}N_{m/0}\mathcal{U}_{\mathfrak{p}_m}) \cap (\mathcal{E}_0\prod_{\mathfrak{p}|p}\mathcal{U}_\mathfrak{p}^{p^m}).$$

Further, all the above isomorphisms are compatible with the natural action of $\mathrm{Gal}(k/\mathbb{Q})$, some of them by class field theory and the others almost trivially. Thus, we obtain an isomorphism

$$I_0 = \varprojlim \mathrm{Gal}(M_{0,m}/L_{0,m}) \simeq \mathcal{V}_0/\mathcal{V}_0 \cap \mathcal{E}_0$$

as $\mathbb{Z}_p[\mathrm{Gal}(k/\mathbb{Q})]$-modules. □

Remark 2. Assume that χ satisfies (A) or (B). In this case, if $A_0(\chi) = \{1\}$, then $A_n(\chi) = \{1\}$ for all $n \geq 0$, and hence $\lambda_p(\chi) = 0$. We give a proof of this more or less known fact for the convenience of readers. Let K_0 be the Hilbert p-class field of k. By Theorem 1 of [3], we have $\mathrm{Gal}(M_0/K_0k_\infty)(\chi) \simeq \mathcal{U}_0(\chi)/\mathcal{E}_0(\chi)$. Therefore, by Lemma 3 and class field theory,

$$\mathrm{Gal}(L_0/k_\infty)(\chi) = \mathrm{Gal}(K_0k_\infty/k_\infty)(\chi) \simeq A_0(\chi).$$

Thus, if $A_0(\chi) = \{1\}$, then, we have $\mathrm{Gal}(L_0/k_\infty)(\chi) = \{1\}$ and hence $\mathrm{Gal}(L/k_\infty)(\chi) = \{1\}$ by Nakayama's lemma. This implies $A_n(\chi) = \{1\}$ for all $n \geq 0$. A related assertion is given in Remark 4(§4).

Finally, let us concentrate on the case (B), i.e., the case $\chi^*(p) = 1$. Let $I'(\chi)$ be the subgroup of $I(\chi)$ isomorphic to

$$\mathbb{T}\mathcal{E}(\chi)/\mathcal{E}(\chi) = \varprojlim \mu_{p^{n+1}}\mathcal{E}_n(\chi)/\mathcal{E}_n(\chi) \tag{18}$$

by the isomorphism in Lemma 3. Then, $I'(\chi) \neq \{1\}$ by Remark 1, and we have an embedding as a Λ-module

$$I'(\chi) \hookrightarrow \Lambda/(T-q)(\simeq \mathbb{Z}_p) \tag{19}$$

with a finite cokernel. Put

$$\mathcal{G}(\chi)_{T-q} = \{g \in \mathcal{G}(\chi) \mid g^{T-q} = 1\}.$$

Lemma 4. *Assume that χ satisfies (B). Then, the following hold.*

(I) $\mathcal{G}(\chi)_{T-q} = I'(\chi)$.

(II) The torsion Λ-module $\mathcal{G}(\chi)/I'(\chi)$ has no nontrivial finite Λ-submodule and

$$\mathrm{char}(\mathcal{G}(\chi)/I'(\chi)) = P_\chi(T)/(T-q) = P_1(T)^{e_1}\cdots P_r(T)^{e_r}. \qquad \blacksquare$$

Here, $P_\chi = (T-q)\prod_i P_i^{e_i}$ is the irreducible decomposition of P_χ in (6).

Proof. Since $\mathcal{G}(\chi)$ has no nontrivial finite Λ-submodule (Fact 2), we have an injective Λ-homomorphism with a finite cokernel;

$$\mathcal{G}(\chi) \hookrightarrow \Lambda/(T-q) \oplus \bigoplus_{i=1}^{r} \bigoplus_{j} \Lambda/(P_i^{e_{i,j}}) \tag{20}$$

for some $e_{i,j}$ with $\sum_j e_{i,j} = e_i (1 \le i \le r)$. From this and $P_i \ne T-q$, we have

$$\mathcal{G}(\chi)_{T-q} \simeq \mathbb{Z}_p.$$

Assume $\mathcal{G}(\chi)_{T-q} \supsetneq I'(\chi)$. Then, by the above,

$$(\mathcal{G}(\chi)_{T-q})^{\omega_0} = (\mathcal{G}(\chi)_{T-q})^p \supseteq I'(\chi).$$

Here, the equality holds because $p \parallel q$ by the definition of q. Therefore

$$I'(\chi)\mathcal{G}(\chi)^{\omega_0}/\mathcal{G}(\chi)^{\omega_0} = \{1\}.$$

On the other hand, the restriction map

$$I(\chi) \to I_0(\chi) \simeq \mathcal{U}_0(\chi)/\mathcal{E}_0(\chi)$$

induces an isomorphism

$$I'(\chi)\mathcal{G}(\chi)^{\omega_0}/\mathcal{G}(\chi)^{\omega_0} \simeq \mu_p\mathcal{E}_0(\chi)/\mathcal{E}_0(\chi).$$

But, the last group is isomorphic to μ_p by Remark 1. This is a contradiction, and hence we obtain $\mathcal{G}(\chi)_{T-q} = I'(\chi)$. The second assertion follows from this, the injectivity of (20) and $T-q \ne P_i$. $\square$

Remark 3. By (3), (4_B) and (19), we immediately obtain $T-q \nmid \operatorname{char} A_\infty(\chi)$, the assertion of ($5_B$).

§4 Proof of Theorem

We assume (C1) and (C2), and use the same notations as in the previous sections.

Fix i with $1 \leq i \leq r$. We easily see from (8),(9),(10) and (14) that the isomorphism in Lemma 2(with $a = a_{i,n}$) maps the class $[c_{i,n}^{Y_{i,n}}]$ in $E'_n/E'^{\,p^a}_n$ to the class $[c_{\chi,n}^{X_{i,n}}]$ in $\mathcal{E}_n/\mathcal{E}_n^{p^a}$. Thus, the condition $(H_{i,n})$ is equivalent to

$$(\mathcal{H}_{i,n}) \qquad c_{\chi,n}^{X_{i,n}} \notin \mathcal{E}_n(\chi)^{p^{a_{i,n}}}.$$

We denote by $(\neg\mathcal{H}_{i,n})$ the opposite of $(\mathcal{H}_{i,n})$:

$$(\neg\mathcal{H}_{i,n}) \qquad c_{\chi,n}^{X_{i,n}} \in \mathcal{E}_n(\chi)^{p^{a_{i,n}}}.$$

Proof of Theorem for the case(A)**.**

Put $Q_i(T) = P_\chi(T)/P_i(T)$ for brevity. By (16) and the Iwasawa main conjecture(Fact 3), we have

$$\operatorname{char} I(\chi) \operatorname{char} A_\infty(\chi) = P_\chi(T).$$

Further, we see that $I(\chi)$ is cyclic over Λ since so is $\mathcal{U}(\chi)$ by (a part of) Fact 1(A) and $I(\chi) \simeq \mathcal{U}(\chi)/\mathcal{E}(\chi)$ as Λ-modules(Lemma 3). Thus, we have $P_i \mid \operatorname{char} A_\infty(\chi)$ if and only if $I(\chi)^{Q_i}$ is finite. But, $I(\chi)^{Q_i}$ is finite only when $I(\chi)^{Q_i} = \{1\}$ because $\mathcal{G}(\chi)$ has no nontrivial finite Λ-submodule(Fact 2). By Lemma 3, we see that $I(\chi)^{Q_i} = \{1\}$ if and only if $\mathcal{U}(\chi)^{Q_i} \subseteq \mathcal{E}(\chi)$. Since the projection $\mathcal{U}(\chi) \to \mathcal{U}_n(\chi)$ is surjective by (11), the latter condition holds if and only if the condition

$$(21_{i,n}) \qquad \mathcal{U}_n(\chi)^{Q_i} \subseteq \mathcal{E}_n(\chi)$$

holds for all $n \geq 0$. So, it suffices to show that $(21_{i,n})$ is equivalent to $(\neg\mathcal{H}_{i,n})$. By Fact 1(A), there is a generator $\mathbf{u}_n$ of $\mathcal{U}_n(\chi)$ over Λ such that $\mathbf{u}_n^{P_\chi} = c_{\chi,n}$. Then, by (7_n) and $\mathbf{u}_n^{\omega_n} = 1$, we observe

$$(22) \qquad c_{\chi,n}^{X_{i,n}} = \mathbf{u}_n^{X_{i,n}P_\chi} = \mathbf{u}_n^{X_{i,n}P_iQ_i} = \mathbf{u}_n^{p^{a_{i,n}}Q_i}.$$

By this, $(21_{i,n})$ is equivalent to $(\neg\mathcal{H}_{i,n})$ since $\mathcal{U}_n(\chi)$ is torsion free over $\mathbb{Z}_p$ (Fact 1(A)). $\square$

Proof for the case(C)**.**

Let Q_i be as before. Similarly to the case(A), we observe that $P_i \mid \operatorname{char} A_\infty(\chi)$ if and only if $\mathcal{U}(\chi)^{Q_i} \subseteq \mathcal{E}(\chi)$. Since the image of the projection $\mathcal{U}(\chi) \to \mathcal{U}_n(\chi)$ is $\mathcal{V}_n(\chi)$, the latter condition holds if and only if the condition

$$(21'_{i,n}) \qquad \mathcal{V}_n(\chi)^{Q_i} \subseteq \mathcal{E}_n(\chi)$$

holds for all $n \geq 1$. We can prove, similarly to the case(A), that $(21'_{i,n})$ is equivalent to $(\neg\mathcal{H}_{i,n})$. Actually, there is a generator $\mathbf{v}_n$ of $\mathcal{V}_n(\chi)$ over Λ such that $\mathbf{v}_n^{P_\chi} = c_{\chi,n}$ by Fact 1(C). Then, we get a formula for this $\mathbf{v}_n$ exactly similar to (22) by using

$$X_{i,n}P_i \equiv p^{a_{i,n}} \bmod \nu_n \quad (\text{by } (7_n)) \text{ and } \mathbf{v}_n^{\nu_n} = 1 (\text{by } (12)). \quad \square$$

Proof for the case(B)**.**

Put $\tilde{P}_\chi(T) = P_\chi(T)/(T-q)$ and $Q_i(T) = \tilde{P}_\chi(T)/P_i(T)$. By (16), (19) and Fact 3, we have

$$\operatorname{char}(I(\chi)/I'(\chi)) \operatorname{char} A_\infty(\chi) = \tilde{P}_\chi.$$

By Lemma 3 and the definition (18) of $I'(\chi)$,

$$(23) \qquad I(\chi)/I'(\chi) \simeq \mathcal{U}(\chi)/\mathbb{T}\mathcal{E}(\chi)$$

as Λ-modules. Then, $I(\chi)/I'(\chi)$ is cyclic over Λ by (a part of) Fact 1(B). Therefore, $P_i \mid \operatorname{char} A_\infty(\chi)$ if and only if $(I(\chi)/I'(\chi))^{Q_i}$ is finite. But, $(I(\chi)/I'(\chi))^{Q_i}$ is finite only when $(I(\chi)/I'(\chi))^{Q_i} = \{1\}$ because $\mathcal{G}(\chi)/I'(\chi)$ has no nontrivial finite Λ-submodule (Lemma 4(II)). By (23), $(I(\chi)/I'(\chi))^{Q_i}$ is trivial if and only if $\mathcal{U}(\chi)^{Q_i} \subseteq \mathbb{T}\mathcal{E}(\chi)$. Because of (11) as in the case (A), the latter condition holds if and only if the condition

$$(21''_{i,n}) \qquad \mathcal{U}_n(\chi)^{Q_i} \subseteq \mathbb{T}_n\mathcal{E}_n(\chi)$$

holds for all $n \geq 0$. It suffices to show that $(21''_{i,n})$ is equivalent to $(\neg\mathcal{H}_{i,n})$. By Fact 1(B), we may take an element $\mathbf{u}_n$ of $\mathcal{U}_n(\chi)$ such that the class $[\mathbf{u}_n]$ generates $\mathcal{U}_n(\chi)/\mathbb{T}_n$ over Λ and that $[\mathbf{u}_n^{\tilde{P}_\chi}] = [c_{\chi,n}]$. Here, for $x \in \mathcal{U}_n(\chi)$, $[x]$ denotes the class in $\mathcal{U}_n(\chi)/\mathbb{T}_n$ represented by x. By (7_n) and $\mathbf{u}_n^{\omega_n} = 1$, we have

$$(22'') \qquad c_{\chi,n}^{X_{i,n}} \equiv \mathbf{u}_n^{p^{a_{i,n}} Q_i} \bmod \mathbb{T}_n$$

similarly to the case (A). Assume $(21''_{i,n})$. Then, by $(22'')$, we have $c_{\chi,n}^{X_{i,n}} \in \mathcal{E}_n(\chi)^{p^{a_{i,n}}}\mathbb{T}_n$. Since $\mathcal{E}_n(\chi)(\ni c_{\chi,n})$ is torsion free over $\mathbb{Z}_p$(by Remark 1), $(\neg\mathcal{H}_{i,n})$ follows from this. Contrarily, if $(\neg\mathcal{H}_{i,n})$ holds, then $(21''_{i,n})$ holds by $(22'')$. □

Remark 4. Assume that χ satisfies (A) or (B). Then, for each $i(1 \leq i \leq r)$, the condition $(H_{i,0})$ holds if and only if $\mathrm{ord}_p P_i(0) > \mathrm{ord}_p \sharp A_0(\chi)$. (In particular, if $A_0(\chi) = \{1\}$, then, $\lambda_p(\chi) = 0$ by Corollary 1). This is proved as follows. We have $a_{i,0} = \mathrm{ord}_p P_i(0)(> 0)$ and $X_{i,0}$ is a unit of $\mathbb{Z}_p$(modulo ω_0) from the definitions of $a_{i,n}$ and $X_{i,n}$. We see that $[\mathcal{U}_0(\chi) : \mathcal{C}_0(\chi)]$ is finite by Fact 1 and (17). Then, by Fact 1 and Remark 1, we get $\mathcal{E}_0(\chi) \simeq \mathbb{Z}_p$. On the other hand, $\sharp A_0(\chi)$ equals to $[\mathcal{E}_0(\chi) : \mathcal{C}_0(\chi)]$ as a consequence of the Iwasawa main conjecture (see [14,Proposition 9]). Thus, $\mathcal{C}_0(\chi)^{X_{i,0}} = \mathcal{C}_0(\chi) = \mathcal{E}_0(\chi)^{\sharp A_0(\chi)}$. The assertion follows from this and $\mathcal{E}_0(\chi) \simeq \mathbb{Z}_p$.

Proof of Lemma 1. Let $n(\geq 0)$ be an integer($n \geq 1$ in the case (C)). It suffices to prove that $(\neg\mathcal{H}_{i,n+1})$ implies $(\neg\mathcal{H}_{i,n})$. Assume that $(\neg\mathcal{H}_{i,n+1})$ holds, namely, that

$$c_{\chi,n+1}^{X_{i,n+1}} \in \mathcal{E}_{n+1}(\chi)^{p^{a_{i,n+1}}}.$$

Taking the norm map, we get

$$c_{\chi,n}^{X_{i,n+1}} \in \mathcal{E}_n(\chi)^{p^{a_{i,n+1}}}.$$

By the definition of $a_{i,n}$, we have $a_{i,n+1} \geq a_{i,n}$. Put $a = a_{i,n+1} - a_{i,n}$. Since $P_i \nmid \omega_n$, we obtain from (7) that

$$X_{i,n+1} \equiv p^a X_{i,n} \bmod \omega_n (\mathrm{resp.} \nu_n)$$

in the case (A) or (B)(resp. (C)). Thus, in the case (A) or (B), we get

$$c_{\chi,n}^{p^a X_{i,n}} \in \mathcal{E}_n(\chi)^{p^a p^{a_{i,n}}} \tag{24}$$

since $c_{\chi,n}^{\omega_n} = 1$. In the case (C), $c_{\chi,n}^{\nu_n} = 1$ by (12) and (15). So, we obtain (24) also in this case. Since $\mathcal{E}_n(\chi)$ has no nontrivial p-th root of unity(Remark 1), (24) implies $(\neg\mathcal{H}_{i,n})$. $\square$

§5 CHARACTERISTIC POLYNOMIAL MODULO A POWER OF p

Let (p,χ) satisfy (C1). We use the same notations as in the previous sections. In this section, we explain a principle for computing the number r of irreducible factors $P_i(T)$ of $P_\chi(T)$, $P_i \bmod p^m$ and the pair $(Y_{i,n}, a_{i,n})$.

Our computation is based upon the following approximation formula ([20,§6]) for the power series $g_\chi(T)(\in \mathbb{Z}_p[[T]])$ related to $L_p(s,\chi)$ by (1). Put

$$\dot{T} = (1+q)(1+T)^{-1} - 1 \text{ and } \dot{\omega}_n = \omega_n(\dot{T}).$$

For an integer a, denote by $\gamma_n(a)$ the unique integer satisfying

$$0 \le \gamma_n(a) < p^n \text{ and } \omega(a)(1+q)^{\gamma_n(a)} \equiv a \bmod p^{n+1}.$$

Then, we have

$$g_\chi(T) \equiv -\frac{1}{2qp^n} \sum_{a=1,(a,q)=1}^{qp^n} a\chi^*(a)^{-1}(1+\dot{T})^{-\gamma_n(a)} \bmod \dot{\omega}_n. \tag{25}$$

Since $g_\chi \in \Lambda \setminus (p)$([10]), $\lambda_\chi^* = \deg P_\chi$ is the minimal n for which the n-th coefficient of g_χ is not divisible by p. So, λ_χ^* is easily calculated from (25).

Further, we need the following two lemmas on p-adic power series and polynomials.

Lemma 5. *Let* $f_1(T)$ *and* $f_2(T)$ *be power series in* $\Lambda \setminus (p)$. *We can write uniquely* $f_i = F_i(T)u_i(T)$ *for a distinguished polynomial* F_i *and a unit* u_i *of* Λ *by the Weierstrass preparation theorem*$(i = 1, 2)$. *Assume that*

$$\deg F_1 = \deg F_2 (:= \lambda) \geq 1.$$

Denote by $l(\geq 1)$ *the minimal natural number satisfying* $\lambda \leq p^l$. *Assume further that*

$$f_1 \equiv f_2 \bmod \dot{\omega}_{z+l} \tag{26}$$

for some $z(\geq 1)$. *Then, we have* $F_1 \equiv F_2 \bmod p^{z+1}$.

Lemma 6. *Let* $F(T)$ *be a distinguished polynomial in* $\mathbb{Z}_p[T]$. *Assume that there exist distinguished polynomials* $G_0(T)$, $H_0(T)$ *such that*

$$F(T) \equiv G_0(T)H_0(T) \bmod\ p^{e+m} \ \textit{and}\ (G_0(T), H_0(T)) \ni p^e$$

for $m \geq e + 1 \geq 1$. *Then there exist distinguished polynomials* $G(T), H(T)$ *satisfying*

$$F(T) = G(T)H(T),\ G(T) \equiv G_0(T) \bmod\ p^m \ \textit{and}\ \ H(T) \equiv H_0(T) \bmod\ p^m.$$

By using the approximation formula for $g_\chi(T)$ and Lemma 5, we can calculate $P_\chi(T) \bmod p^m$ for arbitrary m. In the decomposition (6) of $P_\chi(T)$, it is usually believed that $e_i = 1$ for all i. If $P_\chi \bmod p^m$ has no multiple root for some m, then, we get $e_i = 1$. In the numerical examples in §6, we have checked $e_i = 1$ by this way. Assume $e_i = 1$ for all i. Then, decomposing $P_\chi \bmod p^{m'}$ in $(\mathbb{Z}/p^{m'}\mathbb{Z})[T]$ for some $m'(\gg m)$, we obtain r and $P_i \bmod p^m (1 \leq i \leq r)$ by Lemma 6. For an integer $n(\geq 0)$, using $P_i \bmod p^m$ for "large enough" m, we obtain $(Y_{i,n}, a_{i,n})$ by Euclidean algorithm.

Lemma 6 is known as "Hensel's lemma" and we do not give its proof.

Proof of Lemma 5. The following fact is more or less known(cf. [16,Lemma 4]).

Claim 1 Let $m(\geq 1)$ be an integer. For integers k and j satisfying $0 \leq k \leq m-1$ and $p^k \leq j < p^{k+1}$, the binomial coefficient ${}_{p^m}C_j$ is divisible by p^{m-k}.

Let I be the ideal of Λ generated by p^{z+l+1} and $p^{z+l-s}T^{p^s}(0 \leq s \leq z+l)$. Then, by Claim 1, we easily obtain

$$\dot{\omega}_{z+l} \in I. \tag{27}$$

Let f_1, f_2 be power series satisfying the assumptions of Lemma 5. Clearly,

$$F_1 - F_2 = f_1 u_1^{-1} - f_2 u_2^{-1} = f_1(u_1^{-1} - u_2^{-1}) + (f_1 - f_2)u_2^{-1}.$$

Hence, it suffices to prove

$$f_1 - f_2 \in (p^{z+1}, T^\lambda), \text{ and} \tag{28}$$

$$u_1 - u_2 \in (p^z, T^\lambda) \tag{29}$$

since $f_1 \in (p, T^\lambda)$. The assertion (28) follows from (26), (27) and $\lambda \leq p^l$. Let us prove (29). Writing $f_i = \sum_{j=0}^{\infty} a_j^{(i)} T^j$ with $a_j^{(i)} \in \mathbb{Z}_p$, we put

$$V_i := \sum_{j=\lambda}^{\infty} a_j^{(i)} T^{j-\lambda} \text{ and } R_i := \sum_{j=0}^{\lambda-1} (a_j^{(i)}/p) T^j \quad (i = 1, 2).$$

Then, we have $V_i \in \Lambda^\times$ and $R_i \in \Lambda$ by the first assumption of Lemma 5. Define operators on Λ, τ and $(\tau \cdot h)^j$ with $h \in \Lambda$ and $j \geq 0$, as follows:

$$\tau(\sum_{j=0}^{\infty} b_j T^j) = \sum_{j=\lambda}^{\infty} b_j T^{j-\lambda}$$
$$(\tau \cdot h)^0 = \text{id},$$
$$(\tau \cdot h)^j \cdot f = \tau(h \times ((\tau \cdot h)^{j-1} \cdot f)), \quad f \in \Lambda, \quad j \geq 1.$$

These operators are $\mathbb{Z}_p$-linear. By the last formula of [29,p.114], we have

$$u_i^{-1} = V_i^{-1} \sum_{j=0}^{\infty} (-1)^j p^j (\tau \cdot \frac{R_i}{V_i})^j \cdot 1 \quad (i = 1, 2). \tag{30}$$

Let W be the ideal of Λ generated by p^z and $p^{z-s} T^{p^{l+s}-\lambda} (0 \le s \le z)$. As a $\mathbb{Z}_p$-module, W is generated by

$$\begin{cases} p^z T^j (j < p^{l+1} - \lambda), \\ p^{z-s} T^j (1 \le s \le z-1,\ p^{l+s} - \lambda \le j < p^{l+s+1} - \lambda), \\ T^j (j \ge p^{l+z} - \lambda). \end{cases} \tag{31}$$

For this W, we shall later show the following facts.

Claim 2 $W \subseteq \tau^k(W)$, and $\tau^k(W)$ is an ideal of $\Lambda \quad (k \ge 1)$.

Claim 3 $p^k \tau^k(W) \subseteq (p^z, T^\lambda) \quad (k \ge 1)$.

By (28), $R_1 - R_2 \in p^z \Lambda \subset W$. We also see that $V_1 - V_2 \in W$ from (26) and (27). Hence,

$$\frac{R_1}{V_1} - \frac{R_2}{V_2} = (V_1 V_2)^{-1} \{ (R_1 - R_2) V_2 - R_2 (V_1 - V_2) \} \in W.$$

Therefore, by Claims 2,3 and (30), for proving (29), it suffices to show that

$$(\tau \cdot (f + x))^j \cdot 1 \equiv (\tau \cdot f)^j \cdot 1 \bmod \tau^j(W) \quad (j \ge 1) \tag{32}$$

for all $x \in W$ and $f \in \Lambda$. This is trivial for $j = 1$. Assume that (32) is valid for $j - 1 (\ge 1)$. Then,

$$\begin{aligned} (\tau \cdot (f + x))^j \cdot 1 =& (\tau \cdot (f + x)) \cdot ((\tau \cdot (f + x))^{j-1} \cdot 1) \\ =& (\tau \cdot (f + x)) \cdot ((\tau \cdot f)^{j-1} \cdot 1 + x') \quad (\text{for some } x' \in \tau^{j-1}(W)) \\ =& (\tau \cdot (f + x)) \cdot ((\tau \cdot f)^{j-1} \cdot 1) + \tau((f + x) x') \\ \equiv& (\tau \cdot f)^j \cdot 1 + \tau(x \times (\tau \cdot f)^{j-1} \cdot 1) \bmod \tau^j(W). \end{aligned}$$

The last congruence holds since $\tau^{j-1}(W)$ is an ideal of Λ(Claim 2). But, since $\tau(W) \subset \tau^j(W)$ by Claim 2, we obtain the assertion for j. $\square$

Proof of Claims 2,3. We see from (31) that $\tau^k(W)$ is generated over $\mathbb{Z}_p$ by

$$(33)\quad \begin{cases} p^z T^j \ (j < p^{l+1} - \lambda(k+1), j \geq 0), \\ p^{z-s} T^j \ (1 \leq s \leq z-1, p^{l+s} - \lambda(k+1) \leq j < p^{l+s+1} - \lambda(k+1), j \geq 0) \\ T^j \ (j \geq p^{l+z} - \lambda(k+1), j \geq 0). \end{cases}$$

From this, we easily obtain Claim 2.

Clearly, we have $p^k \cdot p^{z-s} \equiv 0 \bmod p^z$ for s with $s \leq z$ and $s \leq k$, and $p^k \equiv 0 \bmod p^z$ for s with $s > z$ and $s \leq k$. So, looking at (33), we observe that for proving Claim 3, it suffices to show

$$p^{l+s} - \lambda(k+1) \geq \lambda \text{ for any } s > k.$$

If $s > k$, we easily see that $p^s \geq k+2$. The above inequality follows from this and $\lambda \leq p^l$. $\square$

Remark 5. Using the approximation formula (25), several authors have already done some computations on $P_\chi(T)$ or $L_p(s,\chi)$. For example, [28] explains how to compute approximate values of zeros of $P_\chi(T)$ or $L_p(s,\chi)$ and gives some numerical examples.

§6 Tables

In this section, we deal with the real quadratic case. We translate the condition $(H_{P_i,n}) = (H_{i,n})$ into a form more suitable for computer calculation(Lemma 8), and give some numerical result obtained by using the method exploited in this paper. We begin with the following lemma which holds for any (p,χ) satisfying (C1). Put $f' = f$(resp. f/p) when $p \nmid f$(resp. $p \mid f$). Then, $p \nmid f'$ by (C1). For each i with $1 \leq i \leq r$ and $n(\geq 0)$, we put

$$b_{P_i,n} = b_{i,n} = \max(n+1, a_{i,n}).$$

For brevity, we put

$$K_{i,n} = \mathbb{Q}(\mu_{f_n}, \mu_{p^{b_{i,n}}}) = \mathbb{Q}(\mu_{f'p^{b_{i,n}}}).$$

Lemma 7. *$c_{i,n}^{Y_{i,n}} \notin k_n^{\times p^{a_{i,n}}}$ if and only if $c_{i,n}^{Y_{i,n}} \notin K_{i,n}^{\times p^{a_{i,n}}}$.*

This is easily proved in a way similar to Lemma 2 of [17]. By this and the Chebotarev density theorem,we see that the condition $(H_{i,n})$ holds if and only if there is some prime ideal $\mathfrak{L}$ of $K_{i,n}$ of degree one such that

$$c_{i,n}^{Y_{i,n}} \bmod \mathfrak{L} \notin (\mathbb{Z}/l\mathbb{Z})^{\times p^{a_{i,n}}}, \quad l = \mathfrak{L} \cap \mathbb{Q}.$$

Now, let χ be a real quadratic character and k the associated real quadratic field. Write

$$Y_{i,n} = \sum_{j=0}^{p^n-1} a_j(1+T)^j = \sum_{j=0}^{p^n-1} a_j\gamma^j, \quad a_j \in \mathbb{Z}.$$

The integers a_j are determined modulo $p^{a_{i,n}}$. Let τ be the canonical isomorphism $(\mathbb{Z}/f_n\mathbb{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\mu_{f_n})/\mathbb{Q})$. Denote by $\mathfrak{A}_n(\subset \mathbb{Z})$ a complete set of representatives of the subgroup $\tau^{-1}(\mathrm{Gal}(\mathbb{Q}(\mu_{f_n})/k_n))$ of $(\mathbb{Z}/f_n\mathbb{Z})^\times$ and by d an integer such that $\tau_{d|k} \neq id$ and $\tau_{d|\mathbb{Q}_n} = id$. Here, $\mathbb{Q}_n$ is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For a prime number l with $l \equiv 1 \bmod f'p^{b_{i,n}}$ and an integer s satisfying $l \nmid s$ and

$$s \bmod l \text{ is of order } f_n \text{ in } (\mathbb{Z}/l\mathbb{Z})^\times, \tag{34}$$

we put

$$c_i(n,l,s) = \prod_{j,a}\{(1-s^{a(1+q)^j})^{a_j}/(1-s^{ad(1+q)^j})^{a_j}\}^{(t-1)/2}.$$

Here, j runs over all integers with $0 \leq j < p^n$, a runs over $\mathfrak{A}_n$ and $t = p$(resp. p^2) if $\chi(p) \neq -1$ (resp. $\chi(p) = -1$). The rational number $c_i(n,l,s)$ is relatively prime to l. By (34) and $l \equiv 1 \bmod f'p^{b_{i,n}}$, there is a prime ideal $\mathfrak{L}$ of $K_{i,n}$ over l of degree one such that

$$\zeta_{f_n} \equiv s \bmod \mathfrak{L}.$$

Here, ζ_{f_n} is the primitive f_n-th root of unity which appeared in the definition (10) of $c_{i,n}$. Then, for this $\mathfrak{L}$, we easily see that

$$c_{i,n}^{Y_{i,n}} \equiv c_i(n, l, s) \bmod \mathfrak{L}.$$

Denote by $(H_{P_i,n,l})$ the condition that

$$c_i(n, l, s) \notin (\mathbb{Z}/l\mathbb{Z})^{\times p^{a_{i,n}}}$$

holds for some s satisfying (34). From all the above, we obtain

Lemma 8. *Under the above setting, the condition $(H_{P_i,n})$ holds if and only if $(H_{P_i,n,l})$ holds for some prime number l with $l \equiv 1 \bmod f'p^{b_{i,n}}$.*

Remark 6. We can obtain a similar assertion for general (p, χ) satisfying (C1).

Using Lemma 8 and the results of the previous sections, we have obtained, by some computation, the following

Proposition. *$\lambda_3(k) = 0$ for all real quadratic fields $k = \mathbb{Q}(\sqrt{m})$ with m square free and $1 < m < 10^4$.*

Our computation was practiced as follows. Let $p = 3$. For a square free integer $m(> 0)$, put $k = k(m) = \mathbb{Q}(\sqrt{m})$ and let $\chi = \chi_m$ be the associated real Dirichlet character. Put

$$\tilde{P}_\chi(T) = P_\chi (\text{resp. } P_\chi(T)/(T - q))$$

in the case (A) or (C)(resp. (B)), and $\tilde{\lambda}^* = \deg \tilde{P}_\chi$. By (3) and ($5'_B$), $\lambda_3(k) = 0$ if $\tilde{\lambda}^* = 0$. So, assume $\tilde{\lambda}^* \geq 1$. In the range $1 < m < 10^4$, we have seen that

$\tilde{P}_\chi$ has at most two irreducible factors and no multiple roots. Let $P(T)$ be any irreducible factor of $\tilde{P}_\chi$. In the case (A) or (B), we first check the condition $\mathrm{ord}_p P(0) > \mathrm{ord}_p \sharp A_0(\chi)$. If it is satisfied, then, $(H_{P,0})$ holds by Remark 4. If not or in the case (C), we check $(H_{P,1,l})$ for the first 10 prime numbers l with $l \equiv 1 \bmod f'p^{b_{P,1}}$. If it is satisfied for some such l, then, $(H_{P,1})$ holds by Lemma 8. If not, $\cdots$. We have continued this process until $(H_{P,n,l})$ is satisfied for some n and some of the first 10 prime numbers l with $l \equiv 1 \bmod f'p^{b_{P,n}}$. We shall denote this final number n by $\mathbf{n}_P$ in what follows.

By this way, we get the above Proposition. More precisely, we have obtained the following tables for $k = k(m)$ with $1 < m < 10^4$. Table A (resp. Table B, Table C) is for those m's for which χ_m satisfies (A)(resp. (B),(C)). Each table is divided into two parts; one for m's such that $\tilde{P}_\chi$ is irreducible and the other for m's such that $\tilde{P}_\chi$ is reducible. Let us explain how to look at the tables.

Irreducible case

Put $P(T) = \tilde{P}_\chi$. In the block[0](resp.[1]) of Tables A1, B1(resp. Table C1), for each value $\lambda(\geq 0)$, the number M with #-mark after the symbol(λ) means that there are M pieces of m's for which $\mathbf{n}_P = 0$(resp. 1) (in the case $\lambda \geq 1$) and $\tilde{\lambda}^* = \lambda$. In the block[$n$] with $n \geq 1$(resp. $n \geq 2$), for each $\lambda(\geq 1)$ and each integer $m(> 0)$ after (λ), we have seen that the real quadratic field $k(m)$ satisfies $\mathbf{n}_P = n$ and $\tilde{\lambda}^* = \lambda$.

Reducible case

In the range $1 < m < 10^4$, we have seen $\tilde{P}_\chi = P_1P_2$ for some irreducible distinguished polynomials P_1, P_2. The order of P_1, P_2 is given so that $\mathbf{n}_{P_1} \leq \mathbf{n}_{P_2}$. In the block[$n, n'$]($n \leq n'$) in Tables A2, B2, C2, for each pair (λ, λ') of integers λ, $\lambda'(\geq 1)$ and each integer m after the symbol (λ, λ'), we have seen that $k(m)$ satisfies $\mathbf{n}_{P_1} = n$, $\mathbf{n}_{P_2} = n'$ and $\deg P_1 = \lambda$, $\deg P_2 = \lambda'$.

Table A1: Irreducible case.

$[n]$				(λ)	m				
[0]	(0)	#1966	(1)	#587	(2)	#189	(3)	#56	
	(4)	#14							
	(1)	659	761	786	839	894	1091	1101	1191
	1229	1373	1523	1787	1847	1907	1929	2118	2207
	2213	2298	2459	2505	2543	2703	2993	3035	3054
	3062	3221	3261	3281	3602	3719	3873	4106	4193
	4649	4670	4706	4755	4886	4934	4994	5099	5102
	5261	5333	5621	5637	5738	5799	6053	6311	6623
	6686	6782	6807	6809	7058	7226	7259	7262	7319
[1]	7374	7473	7673	7721	7743	7994	8051	8255	8267
	8373	8426	8447	8519	8597	9149	9215	9218	9219
	9278	9293	9413	9419	9467	9551	9902	(2)	1211
	2831	2981	3071	3173	3287	3482	3590	4001	4355
	4367	4853	4982	5042	5255	5798	5918	5930	6401
	6770	6887	7055	7235	7694	8057	8306	8603	8769
	8789	9086	9479	9710	9833	9869	9905	(3)	1406
	3803	4841	8274	8285	9155	(4)	2177	4970	7014
	7019	8999	9830	(6)	5331				
	(1)	443	1758	3594	4098	4215	4238	4481	4511
[2]	4907	5619	5898	6366	7643	7709	7883	8363	8418
	8837	9507	(2)	5529	6995	8742	(3)	3305	5063
	(4)	1937							
[3]	(1)	785	899	2429	2510	3158	3569	4286	7598
	7601	8282	9995	(3)	8711				
[4]	(1)	2666	3047	3846	5081	5297	7658	9590	
[5]	(1)	254	473	1646	6798	6806	7671		
[6]	(1)	9606							

Table A2: Reducible case.

$[n, n']$	(λ, λ') m								
$[0, 0]$	$(1, 1)$	2507	3918	3989	4197	5021	5606	6563	8310
	8465	8555	8994	9165	9326	9777	$(1, 2)$	878	2658
	4241	4427	4526	4661	4737	5690	5993	6141	6254
	7367	9854	$(1, 3)$	8870	$(2, 2)$	7097			
$[0, 1]$	$(1, 1)$	4778	9659	$(1, 2)$	2099	3023			
$[0, 2]$	$(1, 1)$	2918	9578						
$[0, 3]$	$(1, 1)$	8339							
$[1, 1]$	$(1, 1)$	9323	$(1, 2)$	4409	9998	$(1, 3)$	4151		
$[1, 2]$	$(1, 1)$	9813	$(2, 1)$	2021					
$[1, 4]$	$(2, 1)$	9926							

Table B1: Irreducible case.

$[n]$				(λ)	m				
$[0]$	(0)	#497	(1)	#149	(2)	#48	(3)	#5	
	(4)	#3	(5)	#3	(6)	#1	(7)	#1	
	(1)	321	1086	1509	1527	1806	2139	3579	4011
$[1]$	4065	4362	5901	6162	6243	6594	6639	6819	6927
	7566	8277	8403	9897	(2)	906	4434	5595	5991
	7107	8655	(3)	3138	7665	8538	8637	(4)	4641
$[2]$	(1)	3957	7053	8115	9087	9294	9618		
$[3]$	(1)	4749	5613	6414					

Table B2: Reducible case.

$[n, n']$				$(\lambda,$	$\lambda')$	m			
$[0, 0]$	$(1, 1)$	123	915	1518	4794	5829	7467	$(1, 2)$	1095
	7197	7719	$(1, 3)$	7179	8727				
$[0, 1]$	$(1, 1)$	8745	$(1, 2)$	7242	$(1, 3)$	1599			

Table C1: Irreducible case.

$[n]$				(λ)	m				
[1]	(0)	#1444	(1)	#393	(2)	#178	(3)	#36	
	(4)	#15	(5)	#4	(8)	#1			
[2]	(1)	67	106	238	253	454	505	607	610
	787	886	994	1102	1129	1294	1318	1333	1462
	1654	1669	1753	1810	1867	1894	1954	2158	2221
	2371	2410	2419	2515	2521	2593	2737	2743	2971
	3190	3199	3226	3235	3277	3571	3673	3781	3895
	3979	3997	4207	4210	4237	4471	4498	4519	4603
	4615	4618	4651	4681	4711	4867	4870	4954	4963
	5083	5113	5149	5182	5494	5617	5647	5749	5902
	6001	6238	6355	6403	6502	6691	6730	6907	7051
	7078	7294	7387	7522	7603	7621	7633	7639	7711
	7906	7951	8011	8095	8203	8245	8365	8422	8545
	8599	8626	8755	8785	8809	8821	8863	9019	9034
	9103	9115	9145	9202	9463	9754	9766	(2)	2122
	3469	4099	4447	7246	7315	7753	8137	5530	(3)
	6187	9427							
[3]	(1)	295	397	745	1390	2029	2701	2713	3133
	4654	5062	5185	6169	6202	6271	6286	6871	6934
	6955	7957	7969	8017	8155	8569	8782	8965	9058
	9895								
[4]	(1)	3490	4081	7309	7321	7582	9274	9679	
[5]	(1)	1738	2149	3739	4789	5938	8101		
[6]	(1)	2059							
[8]	(1)	9814							

Table C2: Reducible case.

$[n, n']$				$(\lambda,$	$\lambda')$	m			
$[1, 1]$	$(1, 1)$	634	1546	3934	6421	6565	6922	7858	8443
	8665	8761	8974	9745	9811	9982	$(1, 2)$	733	1951
	2914	3958	3973	4729	6751	7438	8413	8581	9799
	$(1, 3)$	4162	6817	$(1, 5)$	4855	$(1, 7)$	5098	$(2, 2)$	7630
$[1, 2]$	$(2, 1)$	5611	5971	$(2, 2)$	9790				
$[1, 3]$	$(2, 1)$	7006	$(3, 1)$	1714					
$[1, 4]$	$(2, 1)$	2917							
$[2, 2]$	$(1, 1)$	2659	$(1, 2)$	8374					
$[2, 3]$	$(1, 1)$	6559	9634						
$[2, 6]$	$(2, 1)$	7726							

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