The first layers of $\mathbf{Z}_p$-extensions and Greenberg's conjecture

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1 Introduction

Let k be a finite extension of $\mathbf{Q}$ and p a prime number. We denote by k_∞ the cyclotomic $\mathbf{Z}_p$-extension of k and by k_n its n-th layer. Let $A_n(k)$ be the p-Sylow subgroup of the ideal class group of k_n. We denote respectively by $\lambda_p(k)$ and $\mu_p(k)$ the Iwasawa λ-invariant and μ-invariant associated to $A_n(k)$. Greenberg proposed a conjecture that $\lambda_p(k) = \mu_p(k) = 0$ for all p and all totally real number fields k (see [8]). As for μ-invariants, Ferrero and Washington proved that $\mu_p(k) = 0$ for all p and all abelian number fields k in [3]. As for λ-invariants, several authors have given sufficient conditions for $\lambda_p(k) = 0$ and verified it for small primes and real quadratic fields with small discriminants (cf. [2], [4], [5], [6], [8], [9], [10], [14], [15], [17], [19], [20]).

In this paper, we study a criterion for $\lambda_p(k) = 0$ given in [2] and [8]. We briefly explain it in a simple case. Let p be an odd prime number, k a real quadratic field and χ the nontrivial Dirichlet character associated to k. Let μ_p be the group of p-th roots of unity and $K = k(\mu_p)$. Assume that p does not split in K. By [11], if $A_0(k)$ is trivial then $\lambda_p(k) = 0$. So we consider the case that $\sharp A_0(k) = p$, where $\sharp A$ is the cardinality of a finite set A. Fix an ideal class $\mathbf{c}$ which generates $A_0(k)$ and denote by $\mathbf{c}_n$ the natural image of $\mathbf{c}$ in $A_n(K)$. Put $\chi^* = \chi^{-1}\omega$, where ω is the Teichmüller character $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}_p$. Let $\mathcal{A}_n = \mathcal{A}_n(\mathbf{c})$ be the set of all elements α_n of K_n satisfying $\mathfrak{a}_n \in \mathbf{c}_n$, $\mathfrak{a}_n^p = (\alpha_n)$ and $\sqrt[p]{\alpha_n} \in M_n(\chi^*)$, where $M_n(\chi^*)$ is the "χ^*-part" of the maximal abelian p-extension of K_n unramified outside p (see §2). In [2]

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and [8], "if part" of the following proposition for $p=3$ and $n=0$ is proved and used for the verification of the conjecture. We will prove it in §3.

Proposition 1. $\lambda_p(k)=0$ *if and only if there is a* $\mathbf{Z}_p$*-extension of* K_n *which contains* $\sqrt[p]{\alpha_n}$ *for some* $\alpha_n \in \mathcal{A}_n$ *and some* $n \geq 0$.

The main purpose of this paper is to determine whether the last statement (J_n) of Proposition 1 holds or not for each n. Put $\Gamma = \mathrm{Gal}(K_\infty/K)$ and denote by γ_0 the topological generator of Γ such that $\zeta^{\gamma_0} = \zeta^{1+p}$ for all p^n-th roots of unity ζ $(n \geq 0)$. Put $A_\infty(k) = \varprojlim A_n(k)$, where the inverse limit is taken with respect to relative norm maps. As usual, a $\mathbf{Z}_p[[\Gamma]]$-module $A_\infty(k)$ can be considered as a $\Lambda = \mathbf{Z}_p[[T]]$-module by the identification $\gamma_0 = 1+T$.

Denote by $g_\chi(T) \in \mathbf{Z}_p[T]$ the Iwasawa polynomial associated to the p-adic L-function $L_p(s,\chi)$ (see details in §2). If the degree of $g_\chi(T)$ is zero, $A_0(k)$ is trivial and the conjecture immediately holds. So, we consider the case that the degree of $g_\chi(T)$ is one. This is a simple but interesting case, because $A_0(k)$ is not always trivial. Then we can uniquely write $g_\chi(T) = T - a_\chi$ for $a_\chi \in p\mathbf{Z}_p$. Moreover there uniquely exists a positive integer or infinity m_χ such that $A_\infty(k)$ is Λ-isomorphic to $\Lambda/(T-a_\chi, p^{m_\chi})$, where we define $p^\infty = 0$. Put $e_\chi = v_p(a_\chi)$, $a_{\chi^*} = \dfrac{p-a_\chi}{1+a_\chi}$ and $e_{\chi^*} = v_p(a_{\chi^*})$, where v_p is the p-adic exponential valuation. In the above setting, we obtain the following theorem.

Theorem 1. *If* $m_\chi = \infty$, (J_n) *does not hold for all* $n \geq 0$. *Otherwise,* (J_n) *holds if and only if* $n \geq m_\chi - \max\{e_\chi, e_{\chi^*}\}$.

By Iwasawa's formula (§6 of [12]), we can easily compute $e_\chi = v_p(L_p(1,\chi))$ and $e_{\chi^*} = v_p(L_p(0,\chi))$. Greenberg's conjecture states that m_χ is finite for all k. Some authors have given efficient methods not only to verify the finiteness of m_χ but also to give an upper bound for m_χ (see [10], [15], [17]). However, when $m_\chi - e_\chi$ is large, it is difficult to compute the real value m_χ.

2 Main theorems

Let p be an odd prime number and K a finite abelian extension of $\mathbf{Q}$ containing μ_p. We fix an inclusion map $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, where $\overline{F}$ is the algebraic closure of a field F. Put $\Delta = \mathrm{Gal}(K/\mathbf{Q})$, $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $G_\infty = \mathrm{Gal}(K_\infty/\mathbf{Q})$. We first assume the following condition.

(C1) The exponent of Δ divides $p-1$.

By (C1), the values of any character ψ of Δ are contained in $\mathbf{Z}_p$. Put $\mathbf{e}_\psi = \dfrac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\psi(\delta)\delta^{-1} \in \mathbf{Z}_p[\Delta]$ the idempotent of the group ring $\mathbf{Q}_p[\Delta]$. For a $\mathbf{Z}_p[\Delta]$-module A, denote by $A(\psi)$ the ψ-component $\mathbf{e}_\psi A$ of A.

As in Introduction, we fix the topological generator γ_0 of Γ such that $\zeta^{\gamma_0} = \zeta^{1+p}$ for all p^n-th roots of unity ζ $(n \geq 0)$. We identify, as usual, the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma_0 = 1 + T$. Thus, for a $\mathbf{Z}_p[[G_\infty]]$-module M, $M(\psi)$ is regarded as a Λ-module, where we identify Δ with $\mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$. Let $A_\infty = A_\infty(K) = \varprojlim A_n(K)$, where the inverse limit is taken with respect to relative norm maps. Then $A_\infty(\psi)$ is regarded as a Λ-module in a natural way.

Let $\mathcal{L}_{(n)}$ be the an abelian extension of K_n and normal over $\mathbf{Q}$. Then, we can consider $\mathrm{Gal}(\mathcal{L}_{(n)}/K_n)$ as a G_∞-module in natural manner. We denote by $\mathcal{L}_{(n)}(\psi)$ the subfield of $\mathcal{L}_{(n)}$ corresponding to $\prod_{\psi' \in \hat{\Delta}, \psi' \neq \psi} \mathrm{Gal}(\mathcal{L}_{(n)}/K_n)(\psi')$ in Galois theory. We denote by $\mathcal{K}_{(n)}$ the composite of all $\mathbf{Z}_p$-extensions of K_n, which is normal over $\mathbf{Q}$.

Let χ be an even character of Δ and we regard χ as a primitive Dirichlet character. Put $\chi^* = \chi^{-1}\omega$, where ω is the Teichmüller character $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}_p$. Assume the following condition.

$$\text{(C2)} \qquad \chi(p) \neq 1 \text{ and } \chi^*(p) \neq 1.$$

Further we set the third assumption concerning the Iwasawa polynomial. Let $L_p(s, \chi)$ be the p-adic L-function associated to χ which is constructed by Kubota and Leopoldt in [16]. By [12], if χ is not the trivial character, there uniquely exists $G_\chi(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\chi((1+p)^{1-s} - 1) = L_p(s, \chi)$ for all $s \in \mathbf{Z}_p$. In [3], it is shown that p does not divide $G_\chi(T)$. Therefore, by p-adic Weierstrass preparation theorem, we can uniquely write $G_\chi(T) = g_\chi(T)u_\chi(T)$, where $g_\chi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\chi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. We call $g_\chi(T)$ the Iwasawa polynomial associated to $L_p(s, \chi)$.

Let M be the maximal abelian p-extension of K_∞ unramified outside p and L the maximal unramified abelian p-extension of K_∞. We denote by M_n (resp. L_n) the maximal abelian extension of K_n in M (resp. L). By the Iwasawa main conjecture proved in [18], the characteristic ideal of $\mathrm{Gal}(M(\chi)/K_\infty)$ (resp. $\mathrm{Gal}(L(\chi^*)/K_\infty)$) is $(g_\chi(T))$ (resp. $(g_\chi(\dot{T}))$), where $(1+T)(1+\dot{T}) = 1+p$. By Theorem 18 of [13], $\mathrm{Gal}(M(\chi)/K_\infty) \simeq \mathrm{Gal}(M/K_\infty)(\chi)$ has no nontrivial finite Λ-submodule. Hence we have an injective Λ-homomorphism

$$Y = \mathrm{Gal}(M(\chi)/K_\infty) \hookrightarrow \bigoplus_{i=1}^{l} \Lambda/(g_i(T)) \qquad (1)$$

with finite cokernel, where $g_\chi(T) = \prod_{i=1}^{l} g_i(T)$ and $g_i(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$.

By class field theory, we have $X = \mathrm{Gal}(L(\chi)/K_\infty) \simeq A_\infty(\chi)$. Hence, if $g_\chi(T) = 1$, then (1) implies that Y and $A_\infty(\chi)$ are trivial. As a next step, we assume the following condition.

(C3) $$A_\infty(\chi) \neq 1 \ \text{ and } g_\chi(T) = T - a_\chi \ \text{ for some } \ a_\chi \in p\mathbf{Z}_p.$$

By (1) and (C3), we have $Y \simeq \Lambda/(T - a_\chi)$. Put $I = \mathrm{Gal}(M(\chi)/L(\chi)) \simeq \mathrm{Gal}(M/L)(\chi)$. Then there exists a positive integer or infinity satisfying

$$I = p^{m_\chi} Y, \text{ i.e. } A_\infty(\chi) \simeq \Lambda/(T - a_\chi, p^{m_\chi}).$$

Fix an ideal class $\mathbf{c}$ which generates $A_0(\chi)[p]$ the p-torsion subgroup of $A_0(\chi)$. Put $\mathcal{A}_n = \{\alpha_n \in K_n | \mathfrak{a}_n \in \mathbf{c}_n,\ \mathfrak{a}_n^p = (\alpha_n) \text{ and } \sqrt[p]{\alpha_n} \in M_n(\chi^*)\}$, where $\mathbf{c}_n$ is the natural image of $\mathbf{c}$ in $A_n(\chi)$. We want to determine whether the following statements hold or not for each $n \geq 0$.

(J_n) $$\sqrt[p]{\alpha_n} \in \mathcal{K}_{(n)}(\chi^*) \text{ for some } \alpha_n \in \mathcal{A}_n.$$

(I_n) $$\sqrt[p]{\alpha_0} \in \mathcal{K}_{(n)}(\chi^*) \text{ for some } \alpha_0 \in \mathcal{A}_0.$$

Put $e_\chi = v_p(a_\chi)$, $a_{\chi^*} = \dfrac{p - a_\chi}{1 + a_\chi}$ and $e_{\chi^*} = v_p(a_{\chi^*})$. Our main results are as follow.

Theorem 1. *Assume* (C1), (C2) *and* (C3). *If* $m_\chi = \infty$, (J_n) *does not hold for all* $n \geq 0$. *Otherwise,* (J_n) *holds if and only if* $n \geq m_\chi - \max\{e_\chi, e_{\chi^*}\}$.

Theorem 2. *Assume* (C1), (C2) *and* (C3). *If* $m_\chi = \infty$, (I_n) *does not hold for all* $n \geq 0$. *If* $m_\chi \leq \max\{e_\chi, e_{\chi^*}\}$, ($\mathrm{I}_n$) *holds for all* $n \geq 0$. *Otherwise,*

$$(\mathrm{I}_n) \textit{ holds if and only if } n \geq \max\{1, m_\chi - e_{\chi^*} + 1\}.$$

In §4, we will prove these theorems.

3 Criteria for Greenberg's conjecture

In this section, we deal with an odd prime p and a real quadratic field k. Then (C1) is satisfied for $K = k(\mu_p)$. Let χ be the nontrivial even character of $\Delta = \mathrm{Gal}(K/\mathbf{Q})$ and $\chi^* = \chi^{-1}\omega$. Put $A_n = A_n(K)$ and $A_\infty = A_\infty(K)$. It is easy to see that $A_n(k) \simeq A_n(\chi)$ since $A_n(\mathbf{Q})$ is trivial for all $n \geq 0$ (see [11]). Assume (C2) and (C3). The prime p does not split in K if and only if (C2) holds.

The following statements (H_n) and (H'_n) hold for some $n \geq 0$ if and only if $\lambda_p(k) = 0$.

(H$_n$) $\quad$ $\mathrm{Gal}(M_n/L_n)(\chi)$ is not trivial.

(H$'_n$) $\quad$ $\mathrm{Ker}(i_{0,n} : A_0 \to A_n)(\chi)$ is not trivial,

where $i_{0,n}$ is induced by the natural inclusion $k_0 \to k_n$.

Using m_χ and e_χ, we can determine whether (H$_n$) (resp. (H$'_n$)) holds or not for each $n \geq 0$ (resp. $n \geq 1$) (see e.g. Proposition 3 of [10]).

Proposition 2. *Let k be a real quadratic field. Assume* (C2) *and* (C3)*. If $m_\chi = \infty$,* (H$_n$) *does not hold for all $n \geq 0$. If $m_\chi < e_\chi$,* (H$_n$) *holds for all $n \geq 0$. Otherwise,*

$$(\mathrm{H}_n) \text{ holds if and only if } n \geq \max\{1, m_\chi - e_\chi + 1\}.$$

For all $n \geq 1$, (H$'_n$) *is equivalent to* (H$_n$)*.*

Remark 1. By Theorem 2 and Proposition 2, we see that (H$_0$) implies (I$_0$)=(J$_0$). But (I$_0$) does not necessarily imply (H$_0$). For example, for $p = 3$ and $k = \mathbf{Q}(\sqrt{443})$, (I$_0$) holds but (H$_0$) does not (see §5). In §4, we see that $p^{e_\chi + n} = \sharp\mathrm{Gal}(M_n/K_\infty)(\chi)$ and $p^{e_{\chi^*} + n} = \sharp\mathrm{Gal}(M_n/K_\infty)(\chi^*)_{tor}$, where A_{tor} is the maximal torsion subgroup of a $\mathbf{Z}_p$-module A. So e_χ (resp. e_{χ^*}) is a important invariant of the χ-part (resp. χ^*-part) of the maximal abelian p-extension of K_∞ unramified outside p. These invariants make an interesting contrast in Theorem 2 and Proposition 2 when m_χ is sufficiently large.

From now, we prove Proposition 1, which give a relation between (J$_n$) and (H$'_n$). Assume (H$'_n$) holds, then for $\alpha_n \in \mathcal{A}_n$ there exists some $\beta_n \in K_n$ such that $(\alpha_n) = (\beta_n^p)$. Therefore, $\sqrt[p]{\alpha_n} \in K_n(\sqrt[p]{E_n}) = K_n(\sqrt[p]{\varepsilon_n} \mid \varepsilon_n \in E_n)$, where $E_n = E_n(K)$ is the group of units of K_n. Conversely, if $\sqrt[p]{\alpha_n} \in K_n(\sqrt[p]{E_n})$, then we have $\alpha_n = \varepsilon_n \beta_n^p$ for some $\varepsilon_n \in E_n$ and $\beta_n \in K_n$. By (C1), $(N_{K_n/k_n}\alpha_n) = (N_{K_n/k_n}\beta_n)^p$ implies (H$'_n$). Therefore we can write

(H$'_n$) $\quad$ $\sqrt[p]{\alpha_n} \in K_n(\sqrt[p]{E_n})(\chi^*)$ for some $\alpha_n \in \mathcal{A}_n$.

Note that

(J$_n$) $\quad$ $\sqrt[p]{\alpha_n} \in \mathcal{K}_{(n)}(\chi^*)$ for some $\alpha_n \in \mathcal{A}_n$.

By the argument of [8, p. 273], if K satisfies (C1) and (C2) we have

$$\bigcup_{n \geq 0} \mathcal{K}_{(n)}(\chi^*) = \bigcup_{m \geq 1} K_m(\sqrt[p^{m+1}]{E_m})(\chi^*). \tag{2}$$

Hence (J$_n$) holds for some $n \geq 0$ if and only if (H$'_m$) holds for some $m \geq 1$. By Theorem 1 of [8], when $\sharp A_0 = p$, (H$'_n$) holds for some $n \geq 1$ if and only if $\lambda_p(k) = 0$. Therefore Proposition 1 follows.

Remark 2. In the above proof, we can write

(H$'_n$) $\qquad\qquad \sqrt[p]{\alpha_0} \in K_n(\sqrt[p]{E_n})(\chi^*)$ for some $\alpha_0 \in \mathcal{A}_0$.

Note that

(I$_n$) $\qquad\qquad \sqrt[p]{\alpha_0} \in \mathcal{K}_{(n)}(\chi^*)$ for some $\alpha_0 \in \mathcal{A}_0$.

By (2), we see that (I$_n$) holds for some $n \geq 0$ if and only if (H$'_m$) holds for some $m \geq 1$.

4 Proof of theorems

Let the notation be the same as in §2. For a $\Lambda' = \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/\mathbf{Q})]]$-module Z, we define a Λ'-module $\dot{Z}$ by

$$\begin{cases} \dot{Z} = Z \text{ as a } \mathbf{Z}_p\text{-module}, \\ \sigma \cdot x = \epsilon(\sigma)\sigma^{-1}x \text{ for } \sigma \in \mathrm{Gal}(K_\infty/\mathbf{Q}) \text{ and } x \in \dot{Z}, \end{cases}$$

where ϵ is the cyclotomic character. Let $E_n(K)$ be the group of units K_n and $E'_n(K)$ the group of p-units of K_n. We put $E(K) = \bigcup_{n \geq 0} E_n(K)$, $E'(K) = \bigcup_{n \geq 0} E'_n(K)$, $N = \bigcup_{n \geq 0} K_n(\sqrt[p^{n+1}]{E_n})$ and $N' = \bigcup_{n \geq 0} K_n(\sqrt[p^{n+1}]{E'_n})$. Further put $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$. Using the arguments of [13], we obtain the following lemma (cf. [8, p. 274]). For convenience of readers, we will give the outline of the proof.

Theorem 3. *Assume* (C1) *and* (C2). *There is an injective Λ'-homomorphism*

$$\mathrm{Gal}(N'(\chi^*)/K_\infty) \hookrightarrow \Lambda'(\chi^*)$$

whose cokernel is Λ'-isomorphic to

$$\mathrm{Hom}(H^1(\Gamma_n, E'(K)), \mathbf{Q}_p/\mathbf{Z}_p)^{\dot{}}(\chi^*) = \mathrm{Hom}(H^1(\Gamma_n, E'(K))(\chi), \mathbf{Q}_p/\mathbf{Z}_p)^{\dot{}}$$

for all sufficiently large n, where $\sigma f(x) = f(\sigma x)$ for $\sigma \in \mathrm{Gal}(K_\infty/\mathbf{Q})$, $f \in \mathrm{Hom}(H^1(\Gamma_n, E'(K)), \mathbf{Q}_p/\mathbf{Z}_p)$, $x \in H^1(\Gamma_n, E'(K))$.

Proof. (Outline) Put $Z = (\mathrm{Gal}(N'(\chi^*)/K_\infty))^{\dot{}} \simeq \mathrm{Gal}(N'/K_\infty)^{\dot{}}(\chi)$ and $D = (E'(K) \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p)(\chi)$. Then we have an orthogonal pairing (cf. [13, p. 280])

$$Z \times D \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Let Z_n denote the annihilator of $(E'_n \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p)(\chi)$ in Z: $Z_n = (E'_n \otimes \mathbf{Q}_p/\mathbf{Z}_p)(\chi)^\perp$.

We have $\omega_n Z = (D^{\Gamma_n})^\perp$, where $\omega_n = (1+T)^{p^n} - 1$ and $D^{\Gamma_n} = \{d \in D \mid d^\gamma = d \text{ for all } \gamma \in \Gamma_n\}$. By (C2), we have

$$(E'_n \otimes \mathbf{Q}_p/\mathbf{Z}_p)(\chi) \simeq (\mathbf{Q}_p/\mathbf{Z}_p)^{p^n} \text{ and } Z/Z_n \simeq \mathbf{Z}_p^{p^n}.$$

By this fact and the argument of [13, p. 281], we have an injective Λ-homomorphism

$$Z \hookrightarrow \Lambda'(\chi)$$

whose cokernel is finite. Put $Z' = \Lambda'(\chi)$ and identify Z with the image of the above map. By the arguments of [13, p. 283],

$$Z'/Z \simeq \omega_n Z'/\omega_n Z = (Z \cap \omega_n Z')/\omega_n Z = Z_n/\omega_n Z$$

for all sufficiently large n. On the other hand, by Lemma 7 of [13], we have

$$D^{\Gamma_n}/(E'_n \otimes \mathbf{Q}_p/\mathbf{Z}_p)(\chi) \simeq H^1(\Gamma_n, E'(K))(\chi).$$

Since $Z_n/\omega_n Z \simeq \mathrm{Hom}(D^{\Gamma_n}/(E'_n \otimes \mathbf{Q}_p/\mathbf{Z}_p)(\chi), \mathbf{Q}_p/\mathbf{Z}_p)$, the theorem follows. □

Remark 3. Let $I_{p,n}$ be the group generated by ideals of K_n all of whose factors lie over p. By (C2), we see that $(I_{p,n} \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p)(\chi)$ is trivial for all n. Hence we have $N(\chi^*) = N'(\chi^*)$, $H^1(\Gamma_n, E(K))(\chi) = H^1(\Gamma_n, E'(K))(\chi)$ and $A_n(\chi) \simeq A'_n(\chi)$, where A'_n is the p-Sylow subgroup of the p-ideal class group of K_n.

In the following, we assume (C1), (C2) and (C3).

Lemma 1. *If m_χ is finite, the order of the maximal $\mathbf{Z}_p$-torsion subgroup of* $\mathrm{Gal}(M_n(\chi^*)/K_n)$ *is* $\min\{m_\chi, n + e_{\chi^*}\}$.

Proof. Let $\mathbf{c}'_n$ be an ideal class which generates $A_n(\chi)[p^{n+1}]$. Then there exists an element α'_n of k_n such that $\mathfrak{c}_n \in \mathbf{c}'_n$, $\mathfrak{c}_n^{p^{n+1}} = (\alpha'_n)$ and $\sqrt[p^{n+1}]{\alpha'_n} \in M_n(\chi^*)$. By Lemma 9 of [13], we have $M(\chi^*) = \bigcup_{n\geq 0} K_n(\sqrt[p^{n+1}]{E_n}, \sqrt[p^{n+1}]{\alpha'_n})(\chi^*)$. Hence the finiteness of m_χ implies $M(\chi^*) = N'(\chi^*)$ (see §3). Therefore by Theorem 3, we have $\mathrm{Gal}(M(\chi^*)/K_\infty) \simeq (\dot{T} - a_\chi, p^{m_\chi}) \simeq (T - a_{\chi^*}, p^{m_\chi})$ as a Λ-module. The maximal $\mathbf{Z}_p$-torsion submodule of $\Lambda/\omega_n(T - \dot{a}_\chi, p^{m_\chi})$ is generated by the class of ω_n, and its order is p^{m_χ}. Since $\langle \omega_n \rangle \cap (T - a_{\chi^*}, p^{m_\chi}) = \langle p^{\max\{0, m_\chi - e_{\chi^*} - n\}} \omega_n \rangle$, the lemma follows. □

Let $\mathfrak{p}$ be a prime ideal of K over p and $\mathfrak{p}_n$ the unique prime ideal of K_n over $\mathfrak{p} = \mathfrak{p}_0$. Denote by $K_{\mathfrak{p}_n}$ the completion of K_n at $\mathfrak{p}_n$ and by $\mathcal{U}_{\mathfrak{p}_n}$ the group of principal units of $K_{\mathfrak{p}_n}$. We put $\mathcal{U}_n = \prod_{\mathfrak{p}|p} \mathcal{U}_{\mathfrak{p}_n}$. Denote by $\mathcal{E}_n$ the closure of the image of $E_n(K)^{q-1}$ under the diagonal embedding $E_n^{q-1} \hookrightarrow \mathcal{U}_n$, where q is the cardinality of the residue class field of a prime ideal of K above p. The Leopoldt conjecture proved in [1] implies that $E_n/E_n^{p^a} \simeq \mathcal{E}_n/\mathcal{E}_n^{p^a}$ for any $a \geq 1$ (see 5.5 of [21]). Put $\mathcal{U} = \varprojlim \mathcal{U}_n$ and $\mathcal{E} = \varprojlim \mathcal{E}_n$, where the inverse limit is taken with respect to relative norm maps. By Theorem 1 of [7], $\mathcal{U}(\chi)$ is Λ-isomorphic to Λ. Fix an isomorphism φ, then we naturally obtain an isomorphism $\varphi_n : \mathcal{U}_n(\chi) \to \Lambda/(\omega_n)$ induced by φ.

Lemma 2. $\varphi_n(\mathcal{E}_n(\chi)) = (T - a_\chi, p^{\max\{0, n-m_\chi+e_\chi\}})/(\omega_n)$.

Proof. By Theorem 2 of [7], $\varphi_n(\mathcal{E}_n)$ includes $(T-a_\chi, \omega_n)/(\omega_n)$. By Lemma 3 (ii′) of [9], we have $\mathcal{U}_n(\chi)/\mathcal{E}_n(\chi) \simeq \mathrm{Gal}(M_n(\chi)/L_n(\chi))$. Here $\mathrm{Gal}(M_n(\chi)/K_\infty) \simeq Y/\omega_n Y \simeq \mathbf{Z}/p^{n+e_\chi}\mathbf{Z}$ and $\mathrm{Gal}(L_n(\chi)/K_\infty) \simeq X/\omega_n X \simeq \mathbf{Z}/p^{\min\{n+e_\chi, m_\chi\}}\mathbf{Z}$, where $Y = \mathrm{Gal}(M(\chi)/K_\infty)$ and $X = \mathrm{Gal}(L(\chi)/K_\infty)$. Hence the lemma follows. □

For $s \geq \max\{n, t_n\}$, put

$$W_{s,n} = \varphi_s(\mathcal{E}_s(\chi))/p^{t_n+1}\varphi_s(\mathcal{E}_s(\chi)),$$

where $t_n = \min\{m_\chi, n + e_{\chi^*}\}$. Further, put $x_{p^s} = [p^{\max\{0, s-m_\chi+e_\chi\}}]$ and $x_i = [T^{p^s-i-1}(T-a)]$ for $1 \leq i \leq p^s - 1$, where $[f(T)]$ is the class of $f(T)$ in $W_{s,n}$. We denote by $W_{s,n}[\dot{\omega}_n]$ the kernel of the map $W_{s,n} \to W_{s,n}$ $(x \mapsto \dot{\omega}_n x)$, where $\dot{\omega}_n = \omega_n(\dot{T}) = (1+\dot{T})^{p^n} - 1$. Since $v_p(\omega_n(a_\chi)) = n + e_\chi$ and $v_p(\omega_n(a_{\chi^*})) = n + e_{\chi^*}$, we can uniquely write $\omega_n(a_\chi) = u_n p^{n+e_\chi}$ and $\omega_n(a_{\chi^*}) = u_n^* p^{n+e_{\chi^*}}$ for $u_n, u_n^* \in \mathbf{Z}_p^\times$.

Lemma 3. *Assume $m_\chi < \infty$ and $s \geq m_\chi - e_\chi$.*
If $n < m_\chi - e_{\chi^*}$, $p^{t_n}(W_{s,n}[\dot{\omega}_n]) \subseteq \langle y_1, y_2, ..., y_{p^n-1}, y_{p^n}\rangle$.
If $n = m_\chi - e_{\chi^*}$, $p^{t_n}(W_{s,n}[\dot{\omega}_n]) \subseteq \langle y_1, y_2, ..., y_{p^n-1}, y_{p^n} - \frac{u_s}{u_n^*} y_{p^s}\rangle$.
If $n > m_\chi - e_{\chi^*}$, $p^{t_n}(W_{s,n}[\dot{\omega}_n]) \subseteq \langle y_1, y_2, ..., y_{p^n-1}, y_{p^s}\rangle$.

Proof. For $1 \leq i \leq p^s$, let c_i be elements of $\mathbf{Z}_p$ such that $[T(T-a_\chi)T^{p^s-2}] = [(T-a_\chi)T^{p^s-1} - \omega_s] = \sum_{i=1}^{p^s} c_i x_i$. Then we easily see that $c_i \in p\mathbf{Z}_p$ and $c_{p^s} = -\omega_s(a_\chi)/p^{s-m_\chi+e}$. Put

$$A = \begin{pmatrix} c_1 & 1 & 0 & ... & 0 & 0 \\ c_2 & 0 & 1 & ... & 0 & 0 \\ .. & .. & .. & ... & .. & .. \\ c_{p^s-2} & 0 & 0 & ... & 1 & 0 \\ c_{p^s-1} & 0 & 0 & ... & 0 & p^{s-m_\chi+e_\chi} \\ c_{p^s} & 0 & 0 & ... & 0 & a_\chi \end{pmatrix}$$

the matrix of (p^s, p^s)-type. Let $A_l = (E+A)^l - (1+p)^l E = (a_{i,j}^{(l)})$, where E is the matrix unit. Since $v_p(c_{p^s}) = m \geq t_n$, we have $a_{p^s,j}^{(l)} \equiv {}_lC_j c_{p^s} \bmod p^{t_n+1}$ for $1 \leq j \leq l$, $a_{p^s,j}^{(l)} = 0$ for $l+1 \leq j \leq p^s - 1$ and $a_{p^s,p^s}^{(l)} = (1+a_\chi)^l - (1+p)^l$. Let $b_1 x_1 + b_2 x_2 + ... + b_{p^s} x_{p^s} \in W_{s,n}[\dot{\omega}_n]$ for $b_j \in \mathbf{Z}_p$. Here we have $\dot{\omega}_n = (1+\dot{T})^{p^n} - 1 = -\dfrac{(1+T)^{p^n} - (1+p)^{p^n}}{(1+T)^{p^n}}$. Since p divides ${}_{p^n}C_j$ for $1 \leq j \leq p^n - 1$, b_{p^n} and b_{p^s} satisfy $c_{p^s} b_{p^n} + ((1+a_\chi)^{p^n} - (1+p)^{p^n}) b_{p^s} \equiv 0 \bmod p^{t_n+1}$. As $(1+a_\chi)^{p^n} - (1+p)^{p^n} = -\left(\dfrac{1+p}{1+a_{\chi^*}}\right)^{p^n} ((1+a_{\chi^*})^{p^n} - 1)$, $p^{m_\chi} u_s b_{p^n} + p^{n+e_{\chi^*}} u_n^* b_{p^s} \equiv$

$0 \bmod p^{t_n+1}$. Since $p^{t_n}(W_{s,n}[\dot{\omega}_n]) \subset \langle y_1, y_2, ..., y_{p^n-1}, y_{p^n}, y_{p^s}\rangle$, this equation implies the lemma. □

We put $\nu_{s,n} = \omega_s/\omega_n$.

Lemma 4. *Assume $m_\chi < \infty$ and $s \geq m_\chi - e_\chi$.*
If $n < m_\chi - e_\chi$, $\langle p^{t_n}[\nu_{s,n}]\rangle = \langle y_{p^n}\rangle$.
If $n = m_\chi - e_\chi$, $\langle p^{t_n}[\nu_{s,n}]\rangle = \langle y_{p^n} + \frac{u_s}{u_n} y_{p^s}\rangle$.
If $n > m_\chi - e_\chi$, $\langle p^{t_n} p^{n-m_\chi+e_\chi}[\nu_{s,n}]\rangle = \langle y_{p^s}\rangle$ and $\langle p^{t_n}[(T - a_\chi)\nu_{s,n}]\rangle = \langle y_{p^n-1}\rangle$.

Proof. If $n \leq m_\chi - e_\chi$, we have $p^{t_n}[\nu_{s,n}] = p^{t_n}[(T^{p^s-p^n-1} + \sum_{j=0}^{p^s-p^n-2} d_j T^j)(T - a_\chi) + \frac{\omega_s(a_\chi)}{\omega_n(a_\chi)}] = [p^{t_n} T^{p^s-p^n-1}(T - a_\chi) + p^{t_n} p^{m_\chi-n-e_\chi} \frac{u_s}{u_n} p^{s-m_\chi+e_\chi}]$, where $d_j \in p\mathbf{Z}_p$. Therefore the first and second assertions follow.

If $n > m_\chi - e_\chi$, we have $p^{t_n} p^{n-m_\chi+e_\chi}[\nu_{s,n}] = p^{t_n} p^{n-m_\chi+e_\chi}[(T^{p^s-p^n-1} + \sum_{j=0}^{p^s-p^n-2} d_j T^j)(T - a_\chi) + \frac{\omega_s(a_\chi)}{\omega_n(a_\chi)}] = [\frac{u_s}{u_n} p^{t_n} p^{s-m_\chi+e_\chi}]$, where $d_j \in p\mathbf{Z}_p$. Further we have $p^{t_n}[(T - a_\chi)\nu_{s,n}] = [p^{t_n}(T - a_\chi)T^{p^s-p^n}]$. Therefore the last assertion follows. □

Let $\mathcal{K}_{(n),k}$ be the composite of all k-th layers of $\mathbf{Z}_p$-extensions of K_n.

Proposition 3. *Assume* (C1), (C2) *and* (C3). *If $m_\chi < \infty$, $\mathcal{K}_{(n),1}(\chi^*) = K_n(\sqrt[p]{E_n})(\chi^*)$ if and only if $n < m_\chi - \max\{e_\chi, e_{\chi^*}\}$ or $n > m_\chi - \min\{e_\chi, e_{\chi^*}\}$.*
If $m_\chi = \infty$, $\mathcal{K}_{(n),1}(\chi^*) = K_n(\sqrt[p]{E_n})(\chi^*)$ for all n.

Proof. Assume $m_\chi < \infty$. Put $D_{s,n} = (E'_s(K)/E'_s(K)^{p^{t_n+1}})(\chi) = (E_s/E_s^{p^{t_n+1}})(\chi)$ and $Z_{s,n} = \mathrm{Gal}(K_s(\sqrt[p^{t_n+1}]{E'_s})(\chi^*)/K_s) \dot{=} \mathrm{Gal}(K_s(\sqrt[p^{t_n+1}]{E_s})(\chi^*)/K_s)$ (see Remark 3). Then we have an orthogonal paring

$$Z_{s,n} \times D_{s,n} \to \mathbf{Z}/p^{t_n+1}\mathbf{Z}. \tag{3}$$

By Lemma 2, we identify $D_{s,n}$ with $W_{s,n}$. By (2), we can take an integer $s = s_n$ such that $K_s(\sqrt[p^{t_n+1}]{E_s})(\chi^*)$ includes $\mathcal{K}_{(n),t_n+1}(\chi^*)$. Then, by (3), the subgroup of $W_{s,n}$ corresponding to $K_s\mathcal{K}_{(n),1}(\chi^*)$ is $p^{t_n}(W_{s,n}[\dot{\omega}_n])$. On the other hand, we have $\varphi_s(i'_{n,s}(u_n)) = [\nu_{s,n}\varphi_n(u_n)] \in \Lambda/(\omega_s)$ for $u_n \in \mathcal{U}_n(\chi)$, where $i'_{n,s}$ is the natural inclusion $\mathcal{U}_n(\chi) \hookrightarrow \mathcal{U}_s(\chi)$. Hence by Lemma 3, the subgroup of $W_{s,n}$ corresponding to $K_s(\sqrt[p]{E_n})(\chi^*)$ is the image of $p^{t_n}\nu_{s,n}(T - a_\chi, p^{\max\{0,n-m_\chi+e_\chi\}})$ in $W_{s,n}$. We have $v_p((1+p)^{p^n}) = n+1$ and $(1+p)^{p^n} = (1+a_\chi)^{p^n}(1+a_{\chi^*})^{p^n} = (1+u_n p^{n+e_\chi})(1+u_n^* p^{n+e_{\chi^*}})$. Hence $u_n \not\equiv -u_n^* \bmod p$ if $e_\chi = e_{\chi^*}(=1)$. Using Lemma 3 and Lemma 4, we obtain the first assertion.

Similarly we can prove the last assertion. Assume $m_\chi = \infty$, then $\mathcal{E}_s(\chi) \simeq \Lambda/(\omega_s)$ for all $s \geq 0$. As $\dot{\omega}_n \equiv -\dfrac{T^{p^n}}{(1+T)^{p^n}} \bmod p$, we have $(\mathcal{E}_s(\chi)^{p^s}/\mathcal{E}_s(\chi)^{p^{s+1}})[\dot{\omega}_n] \simeq (p^sT^{p^s-p^n}, p^{s+1}, \omega_s)/(p^{s+1}, \omega_s)$. Since $p^s\nu_{s,n} \equiv p^sT^{p^s-p^n} \bmod p^{s+1}$, (2) implies the last assertion. □

Let $M_{n,1}$ be the subfield of M_n corresponding to $p\mathrm{Gal}(M_n/K_n)$. Then we have

$$M_{n,1}(\chi^*) = K_n(\sqrt[p]{E_n}, \sqrt[p]{\alpha_n})(\chi^*) \text{ for any } \alpha_n \in \mathcal{A}_n.$$

By the argument of §3, $\sqrt[p]{\alpha_n} \notin K_n(\sqrt[p]{E_n})(\chi^*)$ if $n \leq m_\chi - e_\chi$. Hence for $n \leq m_\chi - e_\chi$, (J_n) holds if and only if $K_s\mathcal{K}_{(n),1}(\chi^*) \neq K_sK_n(\sqrt[p]{E_n})(\chi^*)$. By Proposition 3, we obtain Theorem 1.

Assume (J_0)=(I_0) does not hold. For $n \geq 1$, (I_n) holds if and only if $K_s(\sqrt[p]{E_0}, \sqrt[p]{\alpha_0})(\chi^*) \subseteq K_s\mathcal{K}_{(n),1}$. Here the subgroup of $W_{s,n}$ corresponding to $K_s(\sqrt[p]{E_0}, \sqrt[p]{\alpha_0})(\chi^*)$ is the image of $\langle y_1, y_{p^s}\rangle$. Hence by Lemma 3, we obtain Theorem 2.

5 Numerical examples

Let $p = 3$ and $k = \mathbf{Q}(\sqrt{m})$ with $1 < m < 10^4$ satisfying (C2) and (C3). Put $n_H = n_H(k) = \min\{n|\ (\mathrm{H}_n) \text{ holds for } n\}$, $n_I = n_I(k) = \min\{n|\ (\mathrm{I}_n) \text{ holds for } n\}$ and $n_J = n_J(k) = \min\{n|\ (\mathrm{J}_n) \text{ holds for } n\}$. In the above range, an upper bound m'_χ for m_χ is given for each k (see [10]). On the other hand, by Fukuda's computation (cf. [5]), it is shown that $m_\chi \geq m'_\chi$ for $m \equiv 2 \bmod 3$. Using Proposition 2, Theorem 1 and Theorem 2, we give n_H, n_I and n_J in the following table. For $m \equiv 0 \bmod 3$, the values are upper bounds.

m	e_χ	e_{χ^*}	m_χ	n_H	n_I	n_J
254	1	1	5	5	5	4
257	3	1	1	0	0	0
326	2	1	1	0	0	0
359	2	1	1	0	0	0
443	1	2	2	2	0	0
473	1	1	5	5	5	4
506	2	1	1	0	0	0
659	1	2	1	1	0	0
761	1	1	1	1	0	0
785	1	1	3	3	3	2
786	1	1	1	1	0	0
839	1	1	1	1	0	0
842	2	1	1	0	0	0
894	1	1	1	1	0	0
899	1	1	3	3	3	2
993	2	1	1	0	0	0
1091	1	2	1	1	0	0
1101	1	2	1	1	0	0
1191	1	1	1	1	0	0
1223	2	1	1	0	0	0
1229	1	1	1	1	0	0
1367	2	1	1	0	0	0
1373	1	3	1	1	0	0
1478	2	1	1	0	0	0
1523	1	1	1	1	0	0
1646	1	2	5	5	4	3
1758	1	2	2	2	0	0
1787	1	1	1	1	0	0
1847	1	2	1	1	0	0
1866	3	1	1	0	0	0
1907	1	1	1	1	0	0
1929	1	1	1	1	0	0
2055	2	1	1	0	0	0
2118	1	1	1	1	0	0

m	e_χ	e_{χ^*}	m_χ	n_H	n_I	n_J
2207	1	1	1	1	0	0
2213	1	2	1	1	0	0
2298	1	1	1	1	0	0
2429	1	2	3	3	2	1
2459	1	1	1	1	0	0
2495	3	1	1	0	0	0
2505	1	1	1	1	0	0
2510	1	2	3	3	2	1
2543	1	1	1	1	0	0
2666	1	1	4	4	4	3
2703	1	1	1	1	0	0
2711	3	1	1	0	0	0
2726	2	1	1	0	0	0
2993	1	1	1	1	0	0
3035	1	2	1	1	0	0
3047	1	1	4	4	4	3
3054	1	2	1	1	0	0
3062	1	1	1	1	0	0
3137	3	1	2	0	0	0
3158	1	1	3	3	3	2
3221	1	3	1	1	0	0
3261	1	2	1	1	0	0
3281	1	1	1	1	0	0
3419	2	1	1	0	0	0
3569	1	1	3	3	3	2
3594	1	1	2	2	2	1
3602	1	1	1	1	0	0
3719	2	1	2	1	0	0
3846	1	1	4	4	4	3
3873	1	2	1	1	0	0
3941	2	1	1	0	0	0
3962	2	1	1	0	0	0
3981	2	1	1	0	0	0

m	e_χ	e_{χ^*}	m_χ	n_H	n_I	n_J
4098	1	2	2	2	0	0
4106	1	1	1	1	0	0
4193	1	2	1	1	0	0
4215	1	1	2	2	2	1
4238	1	2	2	2	0	0
4283	2	1	1	0	0	0
4286	1	1	3	3	3	2
4481	1	1	2	2	2	1
4493	2	1	1	0	0	0
4511	1	1	2	2	2	1
4649	1	2	1	1	0	0
4670	1	1	1	1	0	0
4706	1	2	1	1	0	0
4755	1	1	1	1	0	0
4886	1	2	1	1	0	0
4907	1	1	2	2	2	1
4934	1	1	1	1	0	0
4994	1	1	1	1	0	0
5081	1	2	4	4	3	2
5099	1	1	1	1	0	0
5102	1	1	1	1	0	0
5178	2	1	1	0	0	0
5261	1	2	1	1	0	0
5297	1	1	4	4	4	3
5303	2	1	1	0	0	0
5327	2	1	1	0	0	0
5333	1	1	1	1	0	0
5369	2	1	1	0	0	0
5477	2	1	1	0	0	0
5511	3	1	1	0	0	0
5619	1	1	2	2	2	1
5621	1	2	1	1	0	0
5637	1	3	1	1	0	0
5738	1	2	1	1	0	0
5741	2	1	1	0	0	0
5799	1	2	1	1	0	0
5853	2	1	1	0	0	0
5898	1	2	2	2	0	0
5903	3	1	1	0	0	0

m	e_χ	e_{χ^*}	m_χ	n_H	n_I	n_J
6026	3	1	1	0	0	0
6053	1	1	1	1	0	0
6209	2	1	1	0	0	0
6311	1	2	1	1	0	0
6366	1	2	2	2	0	0
6557	4	1	1	0	0	0
6623	1	2	1	1	0	0
6681	3	1	1	0	0	0
6686	1	1	1	1	0	0
6782	1	3	1	1	0	0
6798	1	2	5	5	4	3
6806	1	1	5	5	5	4
6807	1	1	1	1	0	0
6809	2	1	2	1	0	0
6834	2	1	1	0	0	0
7058	1	1	1	1	0	0
7226	2	1	2	1	0	0
7259	1	1	1	1	0	0
7262	1	1	1	1	0	0
7319	1	1	1	1	0	0
7374	1	2	1	1	0	0
7415	2	1	1	0	0	0
7473	1	1	1	1	0	0
7598	1	1	3	3	3	2
7601	1	1	3	3	3	2
7643	1	3	2	2	0	0
7658	1	1	4	4	4	3
7671	1	2	5	5	4	3
7673	1	1	1	1	0	0
7709	1	1	2	2	2	1
7721	1	2	1	1	0	0
7743	1	2	1	1	0	0
7745	2	1	1	0	0	0
7883	1	2	2	2	0	0
7994	1	2	1	1	0	0

m	e_χ	e_{χ^*}	m_χ	n_H	n_I	n_J
8051	1	1	1	1	0	0
8130	2	1	1	0	0	0
8255	1	2	1	1	0	0
8267	1	1	1	1	0	0
8282	1	1	3	3	3	2
8363	1	3	2	2	0	0
8373	1	1	1	1	0	0
8399	2	1	1	0	0	0
8418	1	2	2	2	0	0
8426	1	2	1	1	0	0
8438	2	1	1	0	0	0
8447	1	1	1	1	0	0
8519	1	2	1	1	0	0
8543	4	1	1	0	0	0
8597	1	1	1	1	0	0
8735	5	1	1	0	0	0
8837	1	2	2	2	0	0
8909	2	1	1	0	0	0
8930	3	1	1	0	0	0
9149	1	1	1	1	0	0
9215	1	1	1	1	0	0
9218	1	1	1	1	0	0
9219	1	1	1	1	0	0
9278	1	3	1	1	0	0
9281	2	1	1	0	0	0
9293	1	2	1	1	0	0
9413	1	1	1	1	0	0
9419	1	1	1	1	0	0
9467	1	1	1	1	0	0
9507	1	2	2	2	0	0
9551	1	2	1	1	0	0
9590	1	1	4	4	4	3
9606	2	1	7	6	7	5
9749	2	1	1	0	0	0
9795	2	1	1	0	0	0
9902	1	1	1	1	0	0
9995	1	1	3	3	3	2

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