Isomorphism classes and adjoints of certain Iwasawa modules

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Abstract

Let p be an odd prime number and $\mathcal{O}$ the integer ring of a finite extension of $\mathbf{Q}_p$. We determine isomorphism classes of certain $\mathcal{O}[[T]]$-modules which are isomorphic to $\mathcal{O}^{\oplus 3}$ as $\mathcal{O}$-modules. Moreover we give some examples which are not isomorphic to their adjoints.

1 Results

Let p be an odd prime number and K a finite extension of the field $\mathbf{Q}_p$ of p-adic integers. We denote by $\mathcal{O}$ the integer ring of K and fix a prime element π of $\mathcal{O}$. Put $\Lambda = \mathcal{O}[[T]]$ the ring of one variable formal power series over $\mathcal{O}$.

Let M be a finitely generated torsion Λ-module. By Iwasawa's structure theorem (cf. Chapter 13 of [5]), there is a Λ-homomorphism

$$\varphi : M \to \bigoplus_{i=1}^{l} \Lambda/(f_i(T)) \oplus \bigoplus_{j=1}^{m} \Lambda/(\pi^{\mu_j})$$

with finite kernel and co-kernel, where l, m, $\mu_j \in \mathbf{Z}_{\geq 0}$ and $f_i(T) \in \mathcal{O}[T]$ is a distinguished polynomial. Put $\mathrm{char}(M) = \prod_{i=1}^{l} f_i(T) \prod_{j=1}^{m} \pi^{\mu_j}$.

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For a distinguished polynomial $f(T) \in \mathcal{O}[T]$, we define

$$\mathcal{M}_{f(T)} = \left\{ \Lambda\text{-isomorphism class of } M \;\middle|\; \begin{array}{l} \mathrm{char}(M) = f(T) \text{ and } M \text{ has no} \\ \text{non-trivial finite } \Lambda\text{-submodule} \end{array} \right\}.$$

For all isomorphism classes $[M] \in \mathcal{M}_{f(T)}$, we easily see that $\mathrm{Ker}\ \varphi = 0$ and that $M \cong \mathcal{O}^{\oplus \deg f(T)}$ as $\mathcal{O}$-modules. In [3] and [4], all elements of $\mathcal{M}_{f(T)}$ are determined for all distinguished polynomials $f(T)$ with $\deg f(T) \leq 2$. Further it is shown that $[M] = [\alpha(M)]$ for all $[M] \in \mathcal{M}_{f(T)}$ if $\deg f(T) \leq 2$, where

$$\alpha(M) = \lim_{\leftarrow} \mathrm{Hom}_{\mathcal{O}}(M/\pi^n M, K/\mathcal{O})$$

and $(Ty)(x) = y(Tx)$ for $y \in \mathrm{Hom}_{\mathcal{O}}(M/\pi^n M, K/\mathcal{O})$ and $x \in M/\pi^n M$. Concerning adjoints, see [1] and [2].

In this note, we determine all elements of $\mathcal{M}_{f(T)}$ and their adjoints for $f(T) = (T - \alpha_1)(T - \alpha_2)(T - \alpha_3)$, $\alpha_1,\ \alpha_2,\ \alpha_3 \in (\pi)$, $\alpha_1 \not\equiv \alpha_2 \bmod (\pi^2)$, $\alpha_2 \not\equiv \alpha_3 \bmod (\pi^2)$ and $\alpha_3 \not\equiv \alpha_1 \bmod (\pi^2)$. Put

$$E = \Lambda/(T - \alpha_1) \oplus \Lambda/(T - \alpha_2) \oplus \Lambda/(T - \alpha_3).$$

Denote by $[i, j, a]$ the isomorphism class of the following Λ-submodule of E:

$$\langle (1, 1, 1), (0, \pi^i, a), (0, 0, \pi^j) \rangle.$$

Theorem 1. *Let $f(T)$ be as above. Put $u = \dfrac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1} \in \mathcal{O}^\times$. The cardinality of $\mathcal{M}_{f(T)}$ is seven. Moreover*

$$\mathcal{M}_{f(T)} = \{[0, 0, 0], [0, 1, 0], [1, 0, 0], [0, 1, 1], [1, 2, u\pi], [1, 1, 0], [0, 1, 2]\}.$$

Theorem 2. *Let $f(T)$ be as above. If $[M] \in \mathcal{M}_{f(T)}$ is not $[1, 1, 0]$ nor $[0, 1, 2]$, then*

$$[M] = [\alpha(M)].$$

On the other hand, for $[M_1] = [1, 1, 0]$ and $[M_2] = [0, 1, 2]$,

$$[\alpha(M_1)] = [M_2] \text{ and } [\alpha(M_2)] = [M_1].$$

2 Proofs

In this section, the setting is the same as in Theorem 1 and Theorem 2.

2.1 Proof of Theorem 1

Lemma 1. *For any non-negative integers* m_1, m_2, m_3, *any* $u_1, u_2, u_3 \in \mathcal{O}^\times$ *and submodule* $M \subseteq E$, *a map*

$$\varphi : E \to E \quad ((f_1, f_2, f_3) \mapsto (\pi^{m_1} u_1 f_1, \pi^{m_2} u_2 f_2, \pi^{m_3} u_3 f_3))$$

induces a Λ*-isomorphism* $M \to \varphi(M)$.

Proof. Since π does not divide $f(T)$, φ is an injective map. Therefore the induced map $\varphi : M \to \varphi(M)$ is a Λ-isomorphism. □

Lemma 2. *For any* $u \in \mathcal{O}^\times$ *with* $u \not\equiv 0, 1$, *we have* $[0, 1, u] = [0, 1, 2]$.

Proof. Let $M = \langle (1, 1, 1), (0, 1, u), (0, 0, \pi) \rangle \in [0, 1, u]$. Then for any $u' \in \mathcal{O}$ with $u' \neq u$, there exists some $u'' \in \mathcal{O}^\times$ such that $(u'', 1, u') \in \mathcal{O}^\times(1, 1, 1) + \mathcal{O}(0, 1, u)$. Hence we have $M = \langle (u'', 1, u/2), (0, 1, u), (0, 0, \pi) \rangle$. By Lemma 1, M is Λ-isomorphic to $\langle (1, 1, 1), (0, 1, 2), (0, 0, \pi) \rangle \in [0, 1, 2]$. □

Lemma 3. *For isomorphism classes in* $\mathcal{M}_{f(T)}$, *we have*

$$[0, 0, 0] \ni E,$$

$$\begin{aligned}
[0, 1, 1] &\ni \Lambda/(T - \alpha_1) \oplus \Lambda/((T - \alpha_2)(T - \alpha_3)), \\
[0, 1, 0] &\ni \Lambda/(T - \alpha_2) \oplus \Lambda/((T - \alpha_3)(T - \alpha_1)), \\
[1, 0, 0] &\ni \Lambda/(T - \alpha_3) \oplus \Lambda/((T - \alpha_1)(T - \alpha_2)), \\
[1, 2, u\pi] &\ni \Lambda/((T - \alpha_1)(T - \alpha_2)(T - \alpha_3)).
\end{aligned}$$

Proof. If $\alpha'_1 \not\equiv \alpha'_2 \bmod (\pi^2)$, the cardinality of $\mathcal{M}_{(T-\alpha'_1)(T-\alpha'_2)}$ is two and $\mathcal{M}_{(T-\alpha'_1)(T-\alpha'_2)} = \{[0], [1]\}$, where $[i]$ is the isomorphism class of a submodule $\langle (1, 1), (0, \pi^i) \rangle \subseteq \Lambda/(T - \alpha'_1) \oplus \Lambda/(T - \alpha'_2)$ (cf. [4]). $[0]$ contains $\Lambda/(T - \alpha'_1) \oplus \Lambda/(T - \alpha'_2)$, and $[1]$ contains $\Lambda/((T - \alpha'_1)(T - \alpha'_2))$. We have $\langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle \in [0, 0, 0]$, $\langle (1, 0, 0), (0, 1, 1), (0, 0, \pi) \rangle \in [0, 1, 1]$, $\langle (1, 0, 1), (0, 1, 0), (0, 0, \pi) \rangle \in [0, 1, 0]$ and $\langle (1, 1, 0), (0, \pi, 0), (0, 0, 1) \rangle \in [1, 0, 0]$. Put

$$C = \langle (1, 1, 1), (0, \pi, u\pi), (0, 0, \pi^2) \rangle.$$

Then $C/(\pi, T)C \cong \mathcal{O}/(\pi)$ and C is a cyclic Λ-module. Hence the lemma follows. □

Proof. (Theorem 1) By the structure theorem, for $[M] \in \mathcal{M}_{f(T)}$ there is an injective Λ-homomorphism $\psi : M \to E$ with finite co-kernel. By Lemma 1, we may assume $\psi(M)$ contains $(1,1,1) \in E$. Let $\psi(M) = \langle (1,1,1), (0,\pi^i,a), (0,0,\pi^j)\rangle_{\mathcal{O}}$. Since $TM \subset M$, $\psi(M) \ni (0,\pi,u\pi)$, $(0,0,\pi^2)$. Hence we have $(i,j) = (0,0),(0,1),(1,0),(1,1)$ or $(1,2)$. For $a \equiv a' \bmod (\pi^j)$, we have $[i,j,a] = [i,j,a']$. Therefore, if $(i,j) = (0,0)$, then $M \in [0,0,0]$. In a similar way, if $(i,j) = (1,0)$, then $M \in [1,0,0]$. Since $(0,\pi,u\pi) \in \psi(M)$, if $(i,j) = (1,1)$, π divides a and $M \in [1,1,0]$. Similarly, if $(i,j) = (1,2)$, $M \in [1,2,u\pi]$. By Lemma 2, if $(i,j) = (0,1)$, $M \in [0,1,0]$, $[0,1,1]$ or $[0,1,2]$.

Let $M_1 \in [1,1,0]$ and $M_2 \in [0,1,2]$. For $g(T) \in \Lambda$, denote by $\mathrm{Ker}_{M_1}(g(T))$ (resp. $\mathrm{Ker}_{M_2}(g(T))$) the kernel of the map $M_1 \to M_1$ (resp. $M_2 \to M_2$) $(m \mapsto g(T)m)$. Then

$$\mathrm{Ker}_{M_1}(T-\alpha_k)/((T-\alpha_i)(T-\alpha_j)M_1) \cong \mathcal{O}/(\pi)$$

as $\mathcal{O}$-modules for any $\{i,j,k\} = \{1,2,3\}$. Similarly we have

$$\mathrm{Ker}_{M_2}(T-\alpha_k)/((T-\alpha_i)(T-\alpha_j)M_2) \cong \mathcal{O}/(\pi).$$

Hence M_1 and M_2 are not contained by isomorphism classes which appeared in Lemma 3. Further

$$\mathrm{Ker}_{M_1}((T-\alpha_2)(T-\alpha_3)) \cong \Lambda/(T-\alpha_2) \oplus \Lambda/(T-\alpha_3)$$

and

$$\mathrm{Ker}_{M_2}((T-\alpha_2)(T-\alpha_3)) \cong \Lambda/((T-\alpha_2)(T-\alpha_3)).$$

Therefore $M_1 \ncong M_2$ and this completes the proof. □

2.2 Proof of Theorem 2

Let $[M] \in \mathcal{M}_{f(T)}$ and $M = \langle x_1, x_2, x_3\rangle_{\mathcal{O}}$. Put

$$\delta_{i,j} = \begin{cases} 1 \in K & i = j \\ 0 \in K & i \neq j \end{cases}$$

and

$$y_i = \lim_{\leftarrow} y_{i,n} \in \lim_{\leftarrow} \mathrm{Hom}_{\mathcal{O}}(M/\pi^n M, K/\mathcal{O}),$$

where $y_{i,n} \in \mathrm{Hom}_{\mathcal{O}}(M/\pi^n M, K/\mathcal{O})$ $(y_{i,n}(x_j) = [\delta_{i,j}/\pi^n])$. Then we have $\alpha(M) = \langle y_1, y_2, y_3\rangle_{\mathcal{O}}$. For $g(T) \in \Lambda$, let $A = A(x_1,x_2,x_3,g(T))$ be the $(3,3)$-matrix associated to a transformation $M \to M$ $(m \mapsto g(T)m)$. Then the matrix associated to a transformation $\alpha(M) \to \alpha(M)$ $(m' \mapsto g(T)m')$ is the transposed matrix A^t of A (cf. [3]).

Proof. (Theorem 2)

By Lemma 3 and Corollary 15.27 of [5], the former statement immediately follows.

Let $M_1 = \langle (1,1,1), (0,\pi,0), (0,0,\pi) \rangle \subset E$. Then the matrix associated to a transformation $M_1 \to M_1$ $(m \mapsto (T-\alpha_1)m)$ is

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ (\alpha_2-\alpha_1)/\pi & \alpha_2-\alpha_1 & 0 \\ (\alpha_3-\alpha_1)/\pi & 0 & \alpha_3-\alpha_1 \end{pmatrix}.$$

On the other hand, let $M_2 = \langle (1,1,1), (0,1,2), (0,0,\pi) \rangle \subset E$. Then the matrix associated to a transformation $M_2 \to M_2$ $(m \mapsto (T-\alpha_1)m)$ is

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2-\alpha_1 & \alpha_2-\alpha_1 & 0 \\ (\alpha_1-2\alpha_2+\alpha_3)/\pi & 2(\alpha_3-\alpha_2)/\pi & \alpha_3-\alpha_1 \end{pmatrix}.$$

Put

$$G = \begin{pmatrix} \pi & -\pi & 1 \\ -1 & (1-\pi)/2 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

It is easy to see that

$$G \in GL_3(\mathcal{O}) \text{ and } G^{-1}A_1G = A_2^t.$$

This implies the latter statement. □

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