Examples of the Iwasawa Invariants and the Higher K-groups Associated to Quadratic Fields

Dedicated to Professor Toru Ishihara on his 65th birthday

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Abstract

We compute the Iwasawa invariants of $\mathbf{Q}(\sqrt{f}, \zeta_p)$ in the range $|f| < 200$ and $5 \leq p < 200000$ (resp. $|f| < 10$ and $5 \leq p < 1000000$). These computational results give us concrete information on the higher K-groups of the ring of integers of $\mathbf{Q}(\sqrt{f})$.

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Introduction

Let F be a number field and $\mathcal{O}_F$ the ring of integers of F. Put $K = F(\zeta_p)$ and denote by K_∞ the cyclotomic $\mathbf{Z}_p$-extension of K. Let L_∞ be the maximal unramified abelian p-extension of K_∞ and L'_∞ the maximal unramified abelian p-extension of K_∞ in which every prime divisor lying above p splits completely. Put $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$ and $X'_\infty = \mathrm{Gal}(L'_\infty/K_\infty)$.

It is known that there are relations between Iwasawa modules X'_∞ and Quillen's K-groups $K_n(\mathcal{O}_F)$. The main purpose of this paper is to give concrete information on the Iwasawa invariants of X_∞ and the higher K-groups $K_n(\mathcal{O}_F)$ for quadratic fields F by using these relations.

Following [9, 10], we compute Iwasawa invariants and found some exceptional pairs. Using these pairs, we give exceptional examples of $K_n(\mathcal{O}_F)$. For example, we find that for $5 \leq p < 1000000$, p divides the order of $K_{33588}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})})$ if and only if $p = 7$ or 157229 under the Quillen-Lichtenbaum conjecture.

1 Iwasawa invariants of $\mathbf{Q}(\sqrt{f_\chi}, \zeta_p)$

Let χ be a quadratic Dirichlet character and p an odd prime number. Assume that $\chi \neq \omega^{\frac{p-1}{2}}$, where $\omega = \omega_p$ is the Teichmüller character $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}_p$ such that $\omega(a) \equiv a \bmod p$. Put $F = F_\chi = \mathbf{Q}(\sqrt{f_\chi})$ and $K_n = \mathbf{Q}(\sqrt{f_\chi}, \zeta_{p^{n+1}})$. Let A_n (resp. A'_n) the p-part of the ideal class group (resp. p-ideal class group) of K_n.

Put $G_\infty = \mathrm{Gal}(K_\infty/F)$, $\Delta = \mathrm{Gal}(K_\infty/F_\infty)$ and $\Gamma = \mathrm{Gal}(K_\infty/K)$. Further put $\Delta' = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$ and $e'_\psi = \frac{1}{\sharp\Delta'}\sum_{\delta\in\Delta'}\psi(\delta)\delta^{-1}$ for a Dirichlet character ψ of Δ'. For a $\mathbf{Z}_p[\Delta']$-module A, we denote ${e'}_\psi A$ by A^ψ. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ (resp. $\lambda'_p(\psi)$, $\mu'_p(\psi)$ and $\nu'_p(\psi)$) be the Iwasawa invariants associated to X_∞^ψ (resp. ${X'}_\infty^\psi$), i.e.,

$$\sharp A_n^\psi = p^{\lambda_p(\psi)n+\mu_p(\psi)p^n+\nu_p(\psi)} \quad (\text{resp. } \sharp {A'}_n^\psi = p^{\lambda'_p(\psi)n+\mu'_p(\psi)p^n+\nu'_p(\psi)})$$

for sufficiently large n. By Ferrero-Washington's theorem, we have $\mu_p(\psi) = \mu'_p(\psi) = 0$ for all p and ψ.

Assume that ψ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the p-adic L-function is defined as follows. Let $L_p(s,\psi)$ be the p-adic L-function constructed by [6]. Let f_0 be the least common multiple of f_ψ and p. By [3, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying

$$G_\psi((1+f_0)^{1-s}-1) = L_p(s,\psi)$$

for all $s \in \mathbf{Z}_p$ if $\psi \neq \chi^0$. By [2], it was proved that p does not divide $G_\psi(T)$. Therefore, by the p-adic Weierstrass preparation theorem, we can uniquely write

$$G_\psi(T) = g_\psi(T)u_\psi(T),$$

where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Put $\tilde{\lambda}_p(\psi) = \deg g_\psi(T)$.

For a pair (p,ψ), we assume the following condition

$$\text{(C)} \qquad \psi(p) \neq 1 \text{ and } \psi^{-1}\omega(p) \neq 1.$$

If $\psi(p) \neq 1$, we have $\lambda_p(\psi) = \lambda'_p(\psi)$ and $\nu_p(\psi) = \nu'_p(\psi)$.

We extend the tables of [9, 10] to all primes below 200000.

Proposition 1 *For $|f| < 200$ and $100000 < p < 200000$, all exceptional pairs $(p, \chi\omega^k)$ are given in the following table. The meaning of the symbols are as follows: $[\nu] : \nu(\chi\omega^k) > 0$, $[a_0] : v_p(a_0) > 1$, $[b_0] : v_p(b_0) > 1$, $[\mathrm{lmd}] : \tilde{\lambda}(\chi\omega^k) > 1$, where $a_0 = L_p(1,\chi\omega^k)$ and $b_0 = L_p(0,\chi\omega^k)$.*

<u>Exceptional pairs $(p, \chi\omega^k)$ for $100000 < p < 200000$</u>

f	p	k	f	p	k	f	p	k
	$[\nu]$			$[a_0]$			[lmd]	
8	157229	140434	57	119627	53592	133	185189	119132
28	109829	45474	77	175843	43682	168	104971	21988
56	100937	93200	141	120823	39250	-187	166823	150305
-79	153059	68171	157	150401	101272			
-91	107449	81489		$[b_0]$				
-104	184157	53783	-19	165667	11685			
-119	112241	37701	53	167593	99386			
-120	126691	28093	-71	177473	58993			
149	109211	11960	137	124493	41762			
-183	104803	58845	-152	104399	90165			

Proposition 2 *For* $|f| < 10$, *i.e.,* $f = -3,\ 5,\ -4,\ -7,\ 8$ *or* -8 *and* $200000 < p < 1000000$, *there is only one exceptional pair* $(399181, \chi_{-4}\omega^{1683})$, *which satisfies* $\tilde{\lambda}(\chi_{-4}\omega^{1683}) > 1$.

In Figures 1-2, we compare the actual number of exceptional pairs with the expected number E in the range $200 < p < 200000$.

FIGURE 1. Exceptional pairs (quadratic, 1<|f|<200, 200<p<200000)

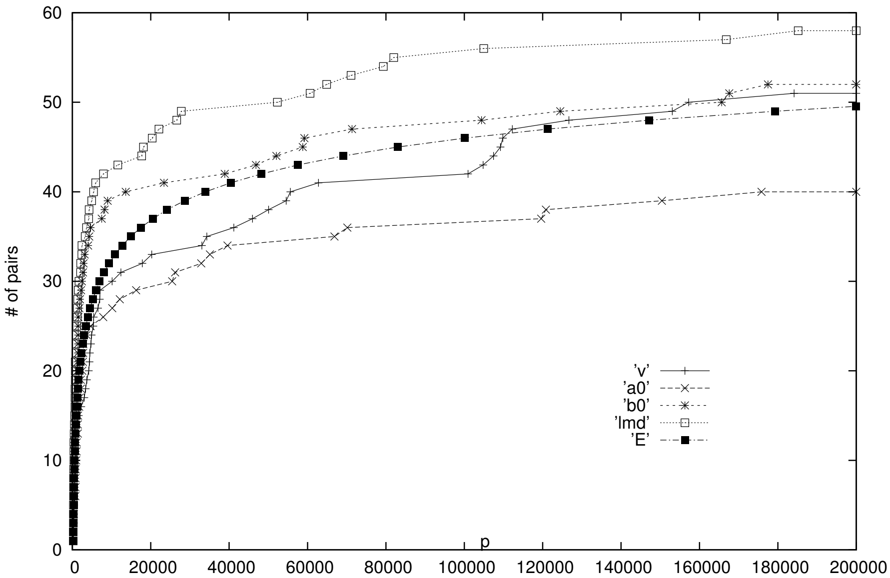
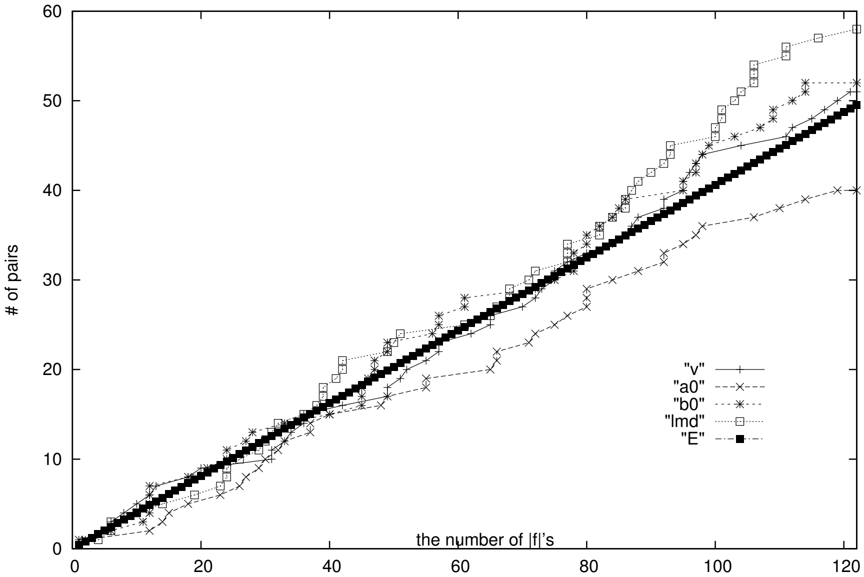


FIGURE 2. Exceptional pairs (Quadratic, 1<|f|<200, 200<p<200000)



From our data, the actual numbers still seem to be near to the expected numbers.

2 Higher K-groups of $\mathcal{O}_F$

We recall some results on Quillen's K-groups.

Theorem 1 (*Quillen*) *For all $n \geq 0$, $K_n(\mathcal{O}_F)$ is a finitely generated* $\mathbf{Z}$*-module.*

Theorem 2 (*Borel*) *For $m \geq 1$,*

$$\mathrm{rank}_{\mathbf{Z}}(K_{2m-1}(\mathcal{O}_F)) = \begin{cases} r_1(F) + r_2(F) & \text{if} \quad m \text{ is odd,} \\ r_2(F) & \text{if} \quad m \text{ is even,} \end{cases}$$

where $r_1(F)$ is the number of real embeddings of F, and $r_2(F)$ is the number of pairs of complex embeddings of F. Further,

$$K_{2m-2}(\mathcal{O}_F) \text{ is finite.}$$

Conjecture 1 (*The Quillen-Lichtenbaum conjecture*) *The natural map* (*via p-adic Chern characters*)

$$K_{2m-i}(\mathcal{O}_F) \otimes \mathbf{Z}_p \to H^i_{\acute{e}t}(Spec(\mathcal{O}_F[1/p]), \mathbf{Z}_p(m))$$

is an isomorphism for all $m \geq 2$, $i = 1, 2$ and any odd prime number p, where $A(m)$ is the m-th Tate twist of a Galois module A.

The surjectivity of p-adic Chern characters was proved by [1, 4, 7, 8]. We simply denote $H^i_{ét}(Spec(\mathcal{O}_F[1/p]), A)$ by $H^i(\mathcal{O}_F, A)$.

Theorem 3 ([5, §3, §4]) *For $m \neq 0$, we have*

$$H^1(\mathcal{O}_F, \mathbf{Z}_p(m))_{tors} \simeq H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(m)).$$

For $m \neq 1$, we have an exact sequence

$$\begin{aligned} 0 \to X'_\infty(m-1)_{G_\infty} \quad &\to H^2(\mathcal{O}_F, \mathbf{Z}_p(m)) \\ &\to \prod_{v|p} H^2(F_v, \mathbf{Z}_p(m)) \to H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(1-m))^\vee \to 0, \end{aligned}$$

where $A^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(A, \mathbf{Q}_p/\mathbf{Z}_p)$.

From now on, we use the same notation as in the previous sections. For an even character $\chi\omega^{1-m}$, we write the Iwasawa polynomial $g_{\chi\omega^{1-m}}(T)$ for the p-adic L-function $L_p(s, \chi\omega^{1-m})$ in the form

$$g_{\chi\omega^{1-m}}(T) = \prod_{i=1}^{\tilde{\lambda}(\chi\omega^{1-m})} (T - \alpha_{\chi\omega^{1-m},i}), \quad \alpha_{\chi\omega^{1-m},i} \in \overline{\mathbf{Q}}_p.$$

We put

$$x(p, \chi, m-1) = \min\{\nu_p(\chi\omega^{1-m}),\ v_p(\prod_{i=1}^{\tilde{\lambda}} \big(1 - (1+f_0)^{m-1}(\alpha_{\chi\omega^{1-m},i} + 1)\big))\}.$$

For an odd character $\chi\omega^{1-m}$, we put $\alpha^*_{\chi\omega^m,i} = \frac{f_0 - \alpha_{\chi\omega^m,i}}{1+\alpha_{\chi\omega^m,i}}$,

$$g^*_{\chi\omega^m}(T) = \prod_{i=1}^{\tilde{\lambda}(\chi\omega^m)} (T - \alpha^*_{\chi\omega^m,i})$$

and

$$x^*(p, \chi, m-1) = v_p(\prod_{i=1}^{\tilde{\lambda}(\chi\omega^m)} \big(1 - (1+f_0)^{m-1}(\alpha^*_{\chi\omega^m,i} + 1)\big)).$$

Further, for an integer m, we define the following sets of prime numbers

$$\begin{aligned} S_1(\chi, m-1) &= \{p : p'|(m-1), (p-1) \nmid (m-1), \chi\omega^{p'}(p) = 1, \chi\omega^{p'} \neq \chi^0\}, \\ S_2(\chi, m-1) &= \{p : (p-1)|(m-1),\ \chi(p) = 1\}, \end{aligned}$$

where $p' = \frac{p-1}{2}$. We put

$$y(p, \chi, m-1) = \begin{cases} v_p(m-1) + 1 & \text{if } p \in S_1(\chi, m-1) \cup S_2(\chi, m-1), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3 *Let χ be a quadratic Dirichlet character, p an odd prime number and $F = F_\chi$. For an odd character $\chi\omega^{1-m}$, if $(p, \chi\omega^m)$ satisfies* (C), *then*

$$\sharp X'_\infty(m-1)^\chi_{G_\infty} = p^{x^*(p,\chi,m-1)}.$$

For an even character $\chi\omega^{1-m}$, assume that $X'^{\chi\omega^{1-m}}_\infty$ is finite. If $(p, \chi\omega^{1-m})$ satisfies (C) *and if $g_{\chi\omega^{1-m}}(T)$ is an Eisenstein polynomial or of degree one, then*

$$\sharp X'_\infty(m-1)^\chi_{G_\infty} = p^{x(p,\chi,m-1)}.$$

Further, for an integer m, we have

$$\frac{\sharp \prod_{v|p} H^2(F_v, \mathbf{Z}_p(m))^\chi}{\sharp H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(1-m))^\chi} = p^{y(p,\chi,m-1)}.$$

Proof. In [9, Proposition 4.1], we prove the above theorem when χ is even. In the same way, using an isomorphism

$$X'_\infty(m-1)^\chi_\Delta \simeq X'^{\chi\omega^{1-m}}_\infty \otimes \mathbf{Z}_p(m-1),$$

we can show the above equations. $\square$

For a positive integer m and a prime number p, we denote by $K_{2m-2}(\mathcal{O}_F)(p)$ the p-Sylow subgroup of $K_{2m-2}(\mathcal{O}_F)$. Here we put

$$K'_{2m-2}(\mathcal{O}_F) = \bigoplus_{5 \le p < 200000} K_{2m-2}(\mathcal{O}_F)(p),$$

$$X'(\chi, m-1) = \prod_{5 \le p < 200000} \sharp X'_\infty(m-1)^\chi_{G_\infty} \text{ and}$$

$$Y'_i(\chi, m-1) = \prod_{p \in S_i(\chi,m),\ 5 \le p < 200000} \frac{\sharp \prod_{v|p} H^2(F_v, \mathbf{Z}_p(m))^\chi}{\sharp H^0(\mathcal{O}_F, \mathbf{Q}_p/\mathbf{Z}_p(1-m))^\chi}.$$

Then, Theorem 3 and the surjectivity of p-adic Chern characters, we have

$$\sharp K'_{2m-2}(\mathcal{O}_F)^\chi \text{ is divided by } X'(\chi, m-1) \cdot Y'_1(\chi, m-1) \cdot Y'_2(\chi, m-1).$$

For an odd character $\chi\omega^{1-m}$, we can compute $v_p(X'(\chi, m-1))$ from the zeros of the Iwasawa polynomial by Proposition 3. In fact, we can easily obtain a lot of examples of (χ, m) with $X'(\chi, m-1) > 1$.

On the other hand, for an even character $\chi\omega^{1-m}$, it is more difficult to obtain examples of (χ, m) with $X'(\chi, m-1) > 1$. Since Vandiver's conjecture is true for all $p < 12000000$, $X'_\infty(m-1)^{\chi^0}_{G_\infty}$ is trivial for any odd integer m. Further we have $\sharp H^2(\mathbf{Q}_p, \mathbf{Z}_p(m)) = \sharp H^0(\mathbf{Q}_p, \mathbf{Q}_p/\mathbf{Z}_p(1-m)) = \sharp H^0(\mathbf{Z}, \mathbf{Q}_p/\mathbf{Z}_p(1-m))$. By Proposition 3 and our computational result [9, 10], we obtain such examples in the following tables.

Factors of $\sharp \mathbf{K}'_{\mathbf{2m-2}}(\mathcal{O}_{\mathbf{Q}(\sqrt{\mathbf{f}_\chi})})$ with $\mathbf{X}' = \mathbf{X}'(\chi, \mathbf{m-1}) > \mathbf{1}$

$-200 < f_\chi < 0$ and $5 \leq p < 200000$

$2m-2$	f_χ	X'	$2m-2$	f_χ	X'
122	-4	379	46	-11	79
22	-11	173	5470	-15	4909
38	-19	37	58	-19	41
594	-19	2251	1714	-20	20261
34	-23	193	30	-31	131
1090	-31	821	26	-40	97
198	-51	557	5918	-51	6553
78178	-55	41189	46	-67	433
26	-71	17	14	-79	17
55534	-79	45943	169774	-79	153059
654	-84	10133	47958	-88	33049
30	-91	37	7550	-91	7069
51918	-91	107449	26	-103	17
102	-103	67	35102	-104	17837
260746	-104	184157	1034	-116	4363
149078	-119	112241	3846	-120	4177
197194	-120	126691	26	-127	67
1450	-131	853	14	-136	11
96258	-136	54547	4434	-139	4451
18	-148	23	490	-152	863
1398	-152	3019	6478	-155	12377
46	-163	79	1102	-167	797
91914	-183	104803	30	-187	79

We have $Y_1'(\chi, m-1) = Y_2'(\chi, m-1) = 1$ for all the above cases.

$1 < f_\chi < 200$ and $5 < p < 200000$

$2m-2$	f_χ	X'	Y_2'	$2m-2$	f_χ	X'	Y_2'
68372	8	34301	1	33588	8	157229	7
316	12	701	1	96	21	199	$5\cdot 17$
128708	28	109829	47	44	33	53	1
20	37	43	11	936	53	1033	$7\cdot 13^2\cdot 37$
15472	56	100937	5	92652	56	55621	$43\cdot 6619\cdot 15443$
8	69	19	5	1220	85	3697	1
88	88	71	1	5124	101	5333	$43\cdot 367$
8	104	19	5	20	113	43	11
3540	113	3373	$7\cdot 11\cdot 31$	140	124	197	11
380	124	239	11	76	129	67	1
9260	140	4751	1	1208	141	5431	5
20	149	43	1	108	149	71	$7\cdot 19$
92	149	229	47	194500	149	109211	251
90936	156	50051	$5\cdot 7\cdot 19$	688	157	401	173
156	161	101	1	28	168	37	1
124	172	73	1	4	173	7	1
20	173	43	1	116	173	101	1
20	177	17	11	36	181	71	1
10724	181	6991	1	944	185	827	1
2904	188	1621	$23\cdot 67\cdot 727$	11380	193	62791	1
296	197	521	1				

We have $Y_1' = Y_1'(\chi, m-1) = 1$ for all the above cases.

Examples
There exist submodules A_i of K-groups such that

$$K_{122}(\mathcal{O}_{\mathbf{Q}(\sqrt{-4})}) \supseteq K'_{122}(\mathcal{O}_{\mathbf{Q}(\sqrt{-4})}) \supseteq A_1 \simeq \mathbf{Z}/(379\mathbf{Z}),$$

$$K_{68372}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq K'_{68372}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq A_2 \simeq \mathbf{Z}/(34301\mathbf{Z}), \text{ and}$$

$$K_{33588}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq K'_{33588}(\mathcal{O}_{\mathbf{Q}(\sqrt{8})}) \supseteq A_3 \simeq \mathbf{Z}/(7 \cdot 157229\mathbf{Z}).$$

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