On the Iwasawa lambda invariant of an imaginary abelian field of conductor $3p^{n+1}$

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Abstract

Let p be an odd prime number with $p \neq 3$, and $K = \boldsymbol{Q}(\cos(2\pi/p), \zeta_3)$. Let K_n be the n-th layer of the cyclotomic $\boldsymbol{Z}_p$-extension over K, and λ_n the Iwasawa lambda invariant of the cyclotomic $\boldsymbol{Z}_3$-extension over K_n. By a theorem of Friedman, it is known that λ_n is stable for sufficiently large n. We prove that when $p \leq 599$, we have $\lambda_n = \lambda_0$ for all $n \geq 1$ with the help of computer. Further, for these p, we calculate the invariant λ_0.

1 Introduction

For a number field N, let λ_N be the Iwasawa lambda invariant of the ideal class group of the cyclotomic $\boldsymbol{Z}_3$-extension $N^{(3)}/N$. Here, for a prime number ℓ, $\boldsymbol{Z}_\ell$ denotes the ring of ℓ-adic integers. Let p be a prime number with $p \neq 3$. We put $\delta = 1$ or 0 according as $p = 2$ or not. Let $K = \boldsymbol{Q}(\cos(2\pi/p^{1+\delta}), \zeta_3)$, and for an integer $n \geq 0$, let K_n be the n-th layer of the cyclotomic $\boldsymbol{Z}_p$-extension over K with $K_0 = K$. Namely, we have

$$K_n = \boldsymbol{Q}(\cos(2\pi/p^{n+1+\delta}), \zeta_3) \quad \text{and} \quad K_n^+ = \boldsymbol{Q}(\cos(2\pi/p^{n+1+\delta})).$$

Here, N^+ denotes the maximal real subfield of an imaginary abelian field N. We put $\lambda_n = \lambda_{K_n}$, $\lambda_n^+ = \lambda_{K_n^+}$ and $\lambda_n^- = \lambda_n - \lambda_n^+$ (≥ 0). It is conjectured that $\lambda_n^+ = 0$ (Greenberg's conjecture). It is known that $\lambda_{n+1}^- \geq \lambda_n^-$ and $\lambda_{n+1}^+ \geq \lambda_n^+$. By a theorem of Friedman [1], the invariant $\lambda_n = \lambda_n^+ + \lambda_n^-$ is stable for sufficiently large n. Recently, we gave in [8] an explicit constant $\boldsymbol{m}_p$ such that λ_n^- is stable for all $n \geq \boldsymbol{m}_p$ using ideas/technique of Horie [5, 6] and some results of cyclotomic Iwasawa theory. Applying this result, we compute the invariant λ_n for all $p \leq 599$ and all n with the help of computer.

Theorem 1 *Let p be a prime number with $5 \leq p \leq 599$. Then we have $\lambda_n = \lambda_0$ for all $n \geq 0$.*

The condition $\lambda_n = \lambda_0$ holds if and only if $\lambda_n^- = \lambda_0^-$ and $\lambda_n^+ = \lambda_0^+$. It is known that

$$0 \leq \lambda_0^+ \leq \lambda_0^- \quad \text{and} \quad 0 \leq \lambda_{n+1}^+ - \lambda_n^+ \leq \lambda_{n+1}^- - \lambda_n^- \tag{1}$$

for $n \geq 0$ (see Greenberg [4, Proposition 1]). Hence the essential part of Theorem 1 is the assertion that $\lambda_n^- = \lambda_0^-$ for all n. Further, we compute the invariants λ_0^+ and λ_0^- for $5 \leq p \leq 599$. Precise results for λ_0^-, Proposition 1 and Table 1, are given in Section 3 and at the end of this paper, respectively. We show that $\lambda_0^+ = 0$ (Proposition 2) and hence obtain the following assertion from Theorem 1.

Theorem 2 *Let p be a prime with $5 \leq p \leq 599$. Then we have $\lambda_n^+ = 0$ for all $n \geq 0$.*

For the case $p = 2$, we show the following

Theorem 3 *Let $p = 2$. Then $\lambda_n = 2$ for all $n \geq 3$ and $\lambda_2 = \lambda_1 = \lambda_0 = 0$. Further, $\lambda_n^+ = 0$ for all $n \geq 0$.*

Remark Let p be an odd prime number. Let L_n be the n-th layer of the cyclotomic $\boldsymbol{Z}_p$-extension over $\boldsymbol{Q}(\cos(2\pi/p), \zeta_4)$. Let λ_n be the lambda invariant of the cyclotomic $\boldsymbol{Z}_2$-extension over L_n. Using [9, Theorem 2], it is shown in [7] that when $p \leq 509$, $\lambda_n = \lambda_0$ for all $n \geq 0$.

2 Lemmas on lambda invariant

First, we introduce some standard notation. Let G be a finite abelian group and χ a $\bar{\boldsymbol{Q}}_3$-valued character of G of (degree one), where $\bar{\boldsymbol{Q}}_3$ is a fixed algebraic closure of the 3-adic rationals $\boldsymbol{Q}_3$. Let $\tilde{\chi}$ be the irreducible character of G over $\boldsymbol{Q}_3$ associated to χ, and $e_{\tilde{\chi}}$ the idempotent of the group ring $\boldsymbol{Q}_3[G]$ associated to $\tilde{\chi}$. For a module M over $\boldsymbol{Q}_3[G]$, we denote by $M(\chi) = e_{\tilde{\chi}}M$ the χ-component of M. We have a canonical decomposition

$$M = \bigoplus_{\chi} M(\chi)$$

where χ runs over a complete set of representatives of the $\boldsymbol{Q}_3$-conjugacy classes of the $\bar{\boldsymbol{Q}}_3$-valued characters of G. Denote by $\boldsymbol{Q}_3(\chi)$ the subfield of $\bar{\boldsymbol{Q}}_3$ generated by the values of χ over $\boldsymbol{Q}_3$. Then the χ-component $M(\chi)$ is naturally regarded as a vector space over $\boldsymbol{Q}_3(\chi)$.

Let $K_n^{(3)}/K_n$ be the cyclotomic $\boldsymbol{Z}_3$-extension, and $K_{n,j}$ the j-th layer of $K_n^{(3)}/K_n$ for $j \geq 0$. Let $A_{n,j}$ be the 3-part of the ideal class group of $K_{n,j}$. Denote by X_n the projective limit of $A_{n,j}$ with respect to the relative norms $K_{n,j+1} \to K_{n,j}$ with $j \geq 0$. We can naturally regard $A_{n-1,j}$ as a subgroup of $A_{n,j}$ since $K_{n,j}/K_{n-1,j}$ is a cyclic extension of degree $p \neq 3$ and $A_{n-1,j}$ is the 3-part of the class group. Actually, it is a direct summand of $A_{n,j}$ (cf. Washington [20, Lemma 16.15]). It follows that X_{n-1} is a direct summand of X_n. We put $Y_n = X_n/X_{n-1}$. Let $\widetilde{X}_n = X_n \otimes \boldsymbol{Q}_3$ and $\widetilde{Y}_n = Y_n \otimes \boldsymbol{Q}_3$ be the tensor products over $\boldsymbol{Z}_3$. By the action of the complex conjugation J, these vector spaces over $\boldsymbol{Q}_3$ are decomposed as

$$\widetilde{X}_n = \widetilde{X}_n^+ \oplus \widetilde{X}_n^- \quad \text{and} \quad \widetilde{Y}_n = \widetilde{Y}_n^+ \oplus \widetilde{Y}_n^-.$$

Then the invariants $\lambda_n^\pm$ and $\lambda_n^\pm - \lambda_{n-1}^\pm$ are nothing but the dimensions of the vector spaces $\widetilde{X}_n^\pm$ and $\widetilde{Y}_n^\pm$ over $\boldsymbol{Q}_3$, respectively. We put $G_n = \mathrm{Gal}(K_n/\boldsymbol{Q})$, $G_n^+ = \mathrm{Gal}(K_n^+/\boldsymbol{Q})$ and $W = \mathrm{Gal}(\boldsymbol{Q}(\zeta_3)/\boldsymbol{Q})$. We can naturally regard $\widetilde{X}_n$ (resp. $\widetilde{X}_n^+$) as a module over $\boldsymbol{Q}_3[G_n]$ (resp. $\boldsymbol{Q}_3[G_n^+]$). Let ω be the character of W representing the Galois action on ζ_3. For a $\bar{\boldsymbol{Q}}_3$-valued character χ of G_n^+, denote by $\chi^* = \omega \times \chi^{-1}$ the character of $G_n = W \times G_n^+$, which is often called the dual of χ. Let λ_{χ^*} and λ_χ be the dimensions of the vector spaces $\widetilde{X}_n(\chi^*)$ and $\widetilde{X}_n^+(\chi)$ over $\boldsymbol{Q}_3(\chi^*) = \boldsymbol{Q}_3(\chi)$, respectively. The reflection theorem [4, Proposition 1] asserts that

$$\lambda_\chi \leq \lambda_{\chi^*}. \tag{2}$$

Let d_χ be the order of χ, and put $d'_\chi = [\boldsymbol{Q}_3(\chi) : \boldsymbol{Q}_3]$. When $n \geq 1$, we have canonical decompositions

$$\widetilde{Y}_n^- = \bigoplus_\chi \widetilde{X}_n(\chi^*) \quad \text{and} \quad \widetilde{Y}_n^+ = \bigoplus_\chi \widetilde{X}_n^+(\chi),$$

where χ runs over a complete set of representatives of the $\boldsymbol{Q}_3$-conjugacy classes of the $\bar{\boldsymbol{Q}}_3$-valued characters of G_n^+ with $p^n | d_\chi$. It follows that

$$\lambda_n^- - \lambda_{n-1}^- = \sum_\chi d'_\chi \cdot \lambda_{\chi^*} \quad \text{and} \quad \lambda_n^+ - \lambda_{n-1}^+ = \sum_\chi d'_\chi \cdot \lambda_\chi. \tag{3}$$

Similarly, we have

$$\lambda_0^- = \sum_{\chi} d'_\chi \cdot \lambda_{\chi^*} \quad \text{and} \quad \lambda_0^+ = \sum_{\chi} d'_\chi \cdot \lambda_\chi \tag{4}$$

where χ runs over a complete set of representatives of the $\boldsymbol{Q}_3$-conjugacy classes of the $\bar{\boldsymbol{Q}}_3$-valued characters of G_0^+. The inequalities (1) are immediate consequences of (2), (3) and (4).

We can naturally regard characters of the Galois groups W and G_n^+ as primitive Dirichlet characters. Then the character ω of W is the Teichmüller character of conductor 3, and the above χ's are even Dirichlet characters of conductor $p^{n+1+\delta}$. For such a character χ, let

$$B_{1,\chi\omega} = \frac{1}{3p^{n+1+\delta}} \sum_{a=1}^{3p^{n+1+\delta}} a\chi\omega(a)$$

be the generalized Bernoulli number. Here, a runs over the integers with $1 \leq a \leq 3p^{n+1+\delta}$ relatively prime to $3p$.

Let $\mathcal{O}_\chi$ be the ring of integers of $\boldsymbol{Q}_3(\chi)$. Iwasawa constructed a power series $g_\chi(T)$ in $\mathcal{O}_\chi[[T]]$ associated to the 3-adic L-function $L_3(s,\,\chi)$ by

$$g_\chi((1+3p^{1+\delta})^s - 1) = L_3(s,\,\chi). \tag{5}$$

(See Chapter 7 of [20].) By the Mazur-Wiles theorem ([16]), λ_{χ^*} equals the lambda invariant of the power series g_χ. In particular, $\lambda_{\chi^*} = 0$ if and only if

$$g_\chi(0) = L_3(0,\,\chi) = -(1-(\chi\omega)(3))B_{1,\,\chi\omega} = -B_{1,\,\chi\omega}$$

is a 3-adic unit. Thus we obtain the following:

Lemma 1 *Let $n \geq 0$ be an integer. For a $\bar{\boldsymbol{Q}}_3$-valued even Dirichlet character χ of conductor $p^{n+1+\delta}$, we have $\lambda_{\chi^*} = 0$ if $B_{1,\chi\omega}$ is relatively prime to* 3.

In this paragraph, let $p \geq 5$. We put $n_0 = \mathrm{ord}_p(3^{p-1}-1)$. Let $\varpi_p = 1$ when 3 is a primitive root modulo p^2, and let

$$\varpi_p = \left(p - 1 - \left[\frac{p}{3}\right]\right) \cdot p^{n_0-1},$$

otherwise. Here, $[x]$ denotes the largest integer $\leq x$. We put

$$\boldsymbol{m}_p = n_0 + \left[\frac{\phi(p-1)\log(3(p-1)\varpi_p)}{\log p}\right],$$

where $\phi(*)$ is the Euler function. The following assertion is a special case of the explicit version of Friedman's theorem given in Theorem 2 and Lemma 1 of [8].

Lemma 2 *Let $p \geq 5$. When $n \geq \boldsymbol{m}_p$, we have $\lambda_{\chi^*} = 0$ for all characters χ of G_n^+ with $p^n | d_\chi$.*

Now, from (3), we obtain the following

Lemma 3 *Let $p \geq 5$. Then we have $\lambda_n^- = \lambda_{\boldsymbol{m}_p - 1}^-$ for any integer $n \geq \boldsymbol{m}_p$.*

In the rest of this section, F denotes a totally real number field. Then the invariant λ_F is conjectured to be 0. Let $A_j = A_j(F)$ be the 3-part of the class group of the j-th layer of the cyclotomic $\boldsymbol{Z}_3$-extension $F^{(3)}/F$. We assume that the prime ideals of F over 3 are fully ramified in $F^{(3)}/F$. The following two lemmas are known to specialists. The first one is a special case of Nakayama's lemma. See for example, Fukuda [2, Theorem 1].

Lemma 4 *If $|A_j(F)| = |A_{j+1}(F)|$ for some j, then $\lambda_F = 0$.*

Let $M/F^{(3)}$ be the maximal unramified pro-3 abelian extension, and $X = X(F) = \mathrm{Gal}(M/F^{(3)})$. We fix a topological generator γ_0 of $\mathrm{Gal}(F^{(3)}/F)$, and consider X as a module over the power series ring $\Lambda = \boldsymbol{Z}_3[[T]]$ in the usual manner with the correspondence $\gamma_0 \leftrightarrow 1 + T$. Denote by $f(T) = f_F(T)$ the characteristic polynomial of the torsion Λ-module X. By definition, we have $\lambda_F = \deg f(T)$. For each $j \geq 0$, let $\nu_j = ((1+T)^{3^j} - 1)/T$. When F is a real abelian field, it is known that $f(T)$ is relatively prime to ν_j for all j by a theorem of Brumer (see pp. 256-266 of Greenberg [3]). Let $v_3(*)$ be the additive valuation on $\bar{\boldsymbol{Q}}_3$ normalized so that $v_3(3) = 1$.

Lemma 5 *Assume further that $f = f_F(T)$ is relatively prime to ν_j for all j. If $\lambda_F = \lambda \geq 1$, then*

$$v_3(|A_j|/|A_0|) \geq \begin{cases} 3^j - 1 & \text{for } 0 \leq j \leq j_\lambda, \\ 3^{j_\lambda} - 1 + \lambda(j - j_\lambda) & \text{for } j > j_\lambda, \end{cases}$$

where j_λ is the largest integer j' satisfying $\lambda \geq 2 \cdot 3^{j'-1}$. Further, when $\lambda = 1$ (resp. ≥ 2), we have $v_3(|A_j|/|A_0|) \geq j$ (resp. $2j$).

Proof. By [20, Lemma 13.15], there exists a Λ-submodule of Y of X such that

$$A_j \simeq X/\nu_j Y$$

for all $j \geq 0$. Let X_{tor} be the maximal $\boldsymbol{Z}_3$-torsion submodule of X, which is a Λ-submodule of X. Put $X' = X/X_{tor}$ and $Y' = YX_{tor}/X_{tor}$. Then, X' and Y' are free $\boldsymbol{Z}_3$-modules, and the characteristic polynomial of Y' equals $f(T) = f_F(T)$. We have

$$|A_j|/|A_0| = |X/\nu_j Y|/|X/Y| = |Y/\nu_j Y| \geq |Y'/\nu_j Y'|. \tag{6}$$

By a similar argument as in the proof of [14, Lemma 8.6], we see that

$$v_3(|Y'/\nu_j Y'|) = v_3\left(\prod_{\zeta}{}' f(\zeta - 1)\right) \tag{7}$$

where in the product $\prod'$, ζ runs over the nontrivial 3^j-th roots of unity. Let α_i $(1 \leq i \leq \lambda)$ be the roots of $f(T) = 0$. Then we have $v_3(\alpha_i) \geq 1/\lambda$ for all i. We see from (6) and (7) that

$$\begin{aligned} v_3(|A_j|/|A_0|) &\geq \sum_{\zeta}{}' \sum_{i=1}^{\lambda} v_3((\zeta - 1) - \alpha_i) \\ &\geq \sum_{\zeta}{}' \sum_{i=1}^{\lambda} \min\{v_3(\zeta - 1), v_3(\alpha_i)\} \\ &\geq \sum_{j'=1}^{j} (3^{j'} - 3^{j'-1})\lambda \min\left\{\frac{1}{2 \cdot 3^{j'-1}}, \frac{1}{\lambda}\right\} \end{aligned} \tag{8}$$

Here, in the sum $\sum'$, ζ runs over the nontrivial 3^j-th roots of unity. When $j > j_\lambda$, we see that the right hand side of (8) equals

$$\sum_{j'=1}^{j_\lambda} (3^{j'} - 3^{j'-1}) + \sum_{j'=j_\lambda+1}^{j} \lambda = 3^{j_\lambda} - 1 + \lambda(j - j_\lambda),$$

and obtain the first assertion in this case. We can show it similarly when $0 \leq j \leq j_\lambda$.

Let us show the second assertion. First, let $\lambda \geq 2$. We have $3^j - 1 \geq 2j$ for all $j \geq 0$. When $j > j_\lambda$, writing $j = j_\lambda + i$ with $i > 0$, we see that

$$3^{j_\lambda} - 1 + \lambda(j - j_\lambda) - 2j = 3^{j_\lambda} - 1 - 2j_\lambda + (\lambda - 2)i \geq 0$$

and obtain the assertion in this case. It is shown similarly for the case $\lambda = 1$.
□

3 Computation of λ_0

3.1 Computation of λ_0^-

Let χ be a $\bar{\boldsymbol{Q}}_3$-valued character of G_0^+. The power series $g_\chi(T) = \sum_{i=0}^{\infty} c_{\chi,i} T^i \in \mathcal{O}_\chi[[T]]$ attached to $L_3(s, \chi)$ by (5) satisfies the congruences

$$g_\chi(T) \equiv -\frac{1}{3^{j+1}p} \sum_{a=1}^{3^{j+1}p} a\chi^*(a)^{-1}(1+T)^{-\gamma_j(a)} \tag{9}$$

modulo $(1+T)^{3^j} - 1$ for $j \geq 0$. (See [20, §7.4].) Here, a runs over the integers with $1 \leq a \leq 3^{j+1}p$ and $(a, 3p) = 1$, and $\gamma_j(a)$ is the integer satisfying $0 \leq \gamma_j(a) < 3^j$ and $(1+3p)^{\gamma_j(a)} \equiv a$ or $-a \bmod 3^{j+1}$ according as $a \equiv 1$ or -1 modulo 3. By a theorem of Ferrero and Washington [20, Theorem 7.15], $g_\chi(T)$ is not divisible by 3. Hence, the λ-invariant λ_{χ^*} of g_χ is the smallest integer i with $c_{\chi,i} \in \mathcal{O}_\chi^\times$. Computing the right-hand side of (9) for $j = 4$, we obtained this value. Then we get the value λ_0^- by (4). All the pairs (p, λ_0^-) with $5 \leq p \leq 599$ and $\lambda_0^- \geq 1$ are given in Table 1 at the end of this paper. We also give in the table the values d_χ, d'_χ and λ_{χ^*} for χ with $\lambda_{\chi^*} \geq 1$. Further, we have

Proposition 1 *For those prime numbers with $5 \leq p \leq 599$ which is not contained in the table, we have $\lambda_0^- = 0$.*

3.2 Computation of λ_0^+

In this section, we prove the following assertion by using a computer.

Proposition 2 *For all primes p with $5 \leq p \leq 599$, we have $\lambda_0^+ = 0$.*

For those primes p with $\lambda_0^- = 0$, we already know $\lambda_0^+ = 0$ by (1). So it suffices to show $\lambda_0^+ = 0$ for the primes p in Table 1. For a real abelian field k, some authors exploited methods to verify $\lambda_k = 0$ by reducing cyclotomic units modulo some prime ideals (cf. [13, 10]), where λ_k is, as before, the

Iwasawa λ-invariant of the cyclotomic $\boldsymbol{Z}_3$-extension $k^{(3)}/k$. They imposed an assumption that the exponent of $\mathrm{Gal}(k/\boldsymbol{Q})$ is two. We explain how to verify $\lambda_k = 0$ without the assumption.

For a subfield k of K_0^+, let k_j be the j-th layer of $k^{(3)}/k$, and $A_j = A_j(k)$ the 3-part of the ideal class group of k_j. For a divisor d of $(p-1)/2$, let $k(d)$ be the unique subfield of K_0^+ with $[k(d) : \boldsymbol{Q}] = d$. Put $d_1 = \mathrm{LCM}\{d_\chi \mid \lambda_{\chi^*} \geq 1\}$. We see that to prove $\lambda_0^+ = 0$, it suffices to show $\lambda_{k(d_1)} = 0$ by (2) and (4).

In some cases, we can prove $\lambda_{k(d_1)} = 0$ (and hence $\lambda_0^+ = 0$) by showing $\lambda_k = 0$ for smaller subfields k. We put $d_2 = 1$ when $|A_0(k(d_1))| = 1$. Otherwise, let d_2 be the smallest divisor of d_1 satisfying $3 \nmid d_1/d_2$ and $|A_0(k(d_2))| = |A_0(k(d_1))|$. We put $d_3 = 1$ when 3 does not split in $k(d_1)/\boldsymbol{Q}$. Otherwise, let d_3 be the smallest divisor of d_1 such that $3 \nmid d_1/d_3$ and the primes over 3 does not split in $k(d_1)/k(d_3)$.

Lemma 6 *We have* $\lambda_0^+ = 0$ *if* $\lambda_{k(d_2)} = \lambda_{k(d_3)} = 0$.

Proof. Let k be a subfield of K_0^+ and $\mathfrak{q}$ a prime ideal of k over 3. Noting that the prime ideals over 3 are fully ramified in $k^{(3)}/k$, denote by $\mathfrak{q}_j$ the unique prime ideal of k_j over $\mathfrak{q}$. Let $k_{j,\mathfrak{q}}$ be the completion of k_j at $\mathfrak{q}_j$, and $U_{j,\mathfrak{q}}$ the group of principal units of $k_{j,\mathfrak{q}}$. For each $j \geq 0$, we put

$$\begin{aligned}
\mathcal{U}_j(k) &= \left\{ (u_{j,\mathfrak{q}}) \in \prod_{\mathfrak{q}|3} U_{j,\mathfrak{q}} \mid \prod_{\mathfrak{q}|3} \left(\frac{u_{j,\mathfrak{q}}, k_{i,\mathfrak{q}}/k_{j,\mathfrak{q}}}{\mathfrak{q}_j} \right) = 1 \ \text{ for all } i \geq j \right\}, \\
\mathcal{V}_j(k) &= \prod_{\mathfrak{q}|3} \Big(\bigcap_{i \geq j} N_{k_{i,\mathfrak{q}}/k_{j,\mathfrak{q}}} U_{i,\mathfrak{q}} \Big) \subseteq \mathcal{U}_j(k), \\
\mathcal{W}_j(k) &= \prod_{\mathfrak{q}|3} \Big(\bigcap_{i \geq j} N_{k_{i,\mathfrak{q}}/k_{j,\mathfrak{q}}} k_{i,\mathfrak{q}}^\times \Big), \\
\mathcal{Z}_j(k) &= \mathcal{U}_j(k)/\mathcal{V}_j(k),
\end{aligned}$$

where $\mathfrak{q}$ runs over the prime ideals of k over 3 and $N_{k_{i,\mathfrak{q}}/k_{j,\mathfrak{q}}}$ denotes the norm map from $k_{i,\mathfrak{q}}$ to $k_{j,\mathfrak{q}}$. Let $E_j' = E_j'(k)$ be the group of 3-units of k_j. Here, we say that an element $x \in k_j^\times$ is a 3-unit when it is relatively prime to $\wp$ for all prime ideals $\wp$ with $\wp \nmid 3$. Embedding E_j' into $\prod_{\mathfrak{q}|3} k_{j,\mathfrak{q}}^\times$ diagonally, denote by $\mathcal{E}_j'(k)$ the topological closure of $\mathcal{U}_j(k) \cap E_j' \mathcal{W}_j(k)$. We see that $\lambda_k = 0$ if and only if (i) the class group $A_0(k)$ capitulates in the cyclotomic $\boldsymbol{Z}_3$-extension $k^{(3)}/k$ and (ii) $\mathcal{U}_j(k) = \mathcal{V}_j(k)\mathcal{E}_j'(k)$ for some j from [19, Theorem 1] (combined with Iwasawa [11, Theorem 12] and [3, Corollary]). It suffices to show that these two conditions are satisfied for $k = k(d_1)$ when $\lambda_{k(d_2)} = \lambda_{k(d_3)} = 0$.

First, assume that $\lambda_{k(d_2)} = 0$. Then the condition (i) is satisfied for

$k = k(d_2)$. From the definition of d_2, we see that the inclusion $k(d_2) \to k(d_1)$ induces an isomorphism $A_0(k(d_2)) \to A_0(k(d_1))$. Therefore, the condition (i) is satisfied for $k = k(d_1)$.

Next, assume that $\lambda_{k(d_3)} = 0$. From the definition of d_3, we can show that the inclusion $k(d_3) \to k(d_1)$ induces an isomorphism $\mathcal{Z}_j(k(d_3)) \to \mathcal{Z}_j(k(d_1))$ for each j using local class field theory, by quite a similar argument as in [18, page 655]. As $\lambda_{k(d_3)} = 0$, the condition (ii) is satisfied for $k = k(d_3)$. By the above isomorphism and $E'_j(k(d_3)) \subseteq E'_j(k(d_1))$, we see that the condition (ii) is satisfied also for $k = k(d_1)$. □

Let E_j be the group of units of k_j, and C_j the group of cyclotomic units of k_j in the sense of Sinnott [17]. We put $B_j = B_j(k) = (E_j/C_j)(3)$, where for a finite abelian group A, $A(3)$ denotes the 3-part of A. We have $|A_j| = |B_j|$ by [17, Theorems 4.1, 5.1] since only one prime number ramifies in $k/\boldsymbol{Q}$. In some cases, the group B_j is easier to handle than A_j

Let L be a set of prime numbers and define the following diagonal map:

$$\phi = \phi_L : E_j \to X_L = \prod_{l \in L} \prod_{\mathcal{L}|l} (\mathcal{O}_{k_j}/\mathcal{L})^\times \qquad (\epsilon \mapsto (\epsilon \bmod \mathcal{L})_{\mathcal{L}|l \in L}),$$

where $\mathcal{O}_{k_j}$ is the ring of integers of k_j. If L satisfies

$$\dim_{\boldsymbol{F}_3} \phi_L(C_j)/\phi_L(C_j)^3 = \operatorname{rank}_{\boldsymbol{Z}} E_j,$$

then the map ϕ induces an isomorphism $B_j \simeq (\phi(E_j)/\phi(C_j))(3)$. As we have an explicit set of generators of C_j, we can obtain that of $\phi(C_j) \bmod X_L^{3^e}$ for any e. Fixing a basis of $X_L/X_L^{3^e}$ over $\boldsymbol{Z}/3^e$, if we find that $\phi(C_j) \bmod X_L^{3^e}$ is generated, for instance, by the vectors

$$(1, *, *, *, *),\ (0, 1, *, *, *),\ (0, 0, 3^r, 3^r a, 3^r b),\ (0, 0, 0, 3^s, 3^s c) \qquad (10)$$

for some $1 \le r \le s < e$ and integers a, b, $c \in \boldsymbol{Z}$, then we see that B_j is isomorphic to a subgroup of $B'_j = \boldsymbol{Z}/3^r \oplus \boldsymbol{Z}/3^s$ and that the value $|B'_j| = 3^{r+s}$ is an upper bound of $|B_j| = |A_j|$. Choosing some L such that $|L| \le 5$ and $\ell \equiv 1 \bmod 3^{j+1}p$ for all $\ell \in L$, we computed a set of generators of $\phi(C_j)$ modulo X_L^{3e} for some e in a form similar to (10) by elementary row operation, and obtained an explicit abelian group $B'_j = B'_j(k)$ containing a subgroup isomorphic to B_j. We call B'_j an "upper bound" of B_j.

Table 4 deals with prime numbers p such that $5 \le p \le 599$ and $\lambda_0^- \ge 1$. In the table, we give the values of d_1, d_2 and d_3, and an upper bound

$B'_j = B'_j(k(d))$ for $d = d_2$, d_3. The vector $(q_{j,1}, \cdots, q_{j,r_j})$ in the row B'_j shows that $B'_j = \boldsymbol{Z}/q_{j,1} \oplus \cdots \oplus \boldsymbol{Z}/q_{j,r_j}$. The symbol (-) in the row B'_j means that $B'_j = B'_{j-1}$.

Proof of Proposition 2. By Lemma 6, it is sufficient to check $\lambda_{k(d_2)} = \lambda_{k(d_3)} = 0$ for all prime numbers p with $5 \leq p \leq 599$ and $\lambda_0^- \geq 1$.

After Schoof's computation [20, pp. 420-423], Koyama and Yoshino [12] gave the l-rank of the ideal class group of $\boldsymbol{Q}(\zeta_p)^+$ with $3 \leq p < 10000$ and $2 \leq l \leq 10000$. According to their table, there are four prime numbers p in our cases with $|A_0(k(d_1))| > 1$; $(p, d_1) = (229, 6)$, $(257, 2)$, $(401, 8)$, and $(521, 26)$. For these primes, we checked upper bounds for $|A_0(k(d_1))|$ by the above method. It turned out that they equal the lower bounds which are obtained from [12]. Thus, we could determine the exact value of $|A_0(k(d_1))| = |B_0(k(d_1))|$. By the definition of d_2, we have $|A_0(k(d_2))| = |A_0(k(d_1))|$. For $p = 229$, we see that $|A_2(k(d_2))| \leq 9$ from Table 2 and hence $\lambda_{k(d_2)} = 0$ by Lemma 5. It is shown similarly for $p = 257$. For $p = 401$ and 521, we see that $|A_1(k(d_2))| = |A_0(k(d_2))|$ from Table 2 and hence $\lambda_{k(d_2)} = 0$ by Lemma 4. For the other pairs (p, d_1), we have $k(d_2) = \boldsymbol{Q}$ as $|A_0(k(d_1))| = 1$, and hence $\lambda_{k(d_2)} = 0$ by [20, Theorem 10.4].

We see that $\lambda(k(d_3)) = 0$ except for $p = 271$ and 523 from Table 2 and Lemma 5. Let us show the assertion for $p = 271$ and 523. Let $X = X(k(d_3))$ and $f(T)$ the characteristic polynomial of the Λ-module X. By definition, we have $\lambda_{k(d_3)} = \deg f(T)$ $(= \lambda)$. We put

$$G(T) = \prod_{\chi} g_{\chi}(T)$$

where χ runs over *all* the $\bar{\boldsymbol{Q}}_3$-valued characters of G_0^+ with $\lambda_{\chi^*} \geq 1$. We see that $G(T) \in \boldsymbol{Z}_3[[T]]$ since all conjugates over $\boldsymbol{Q}_3$ of one χ appear in the product. Let γ_0 be the topological generator of $\mathrm{Gal}(k^{(3)}/k) = \mathrm{Gal}(K^{(3)}/K)$ which we fixed before, and choose $c \in 1 + 3\boldsymbol{Z}_3$ so that $\zeta^{\gamma_0} = \zeta^c$ for all 3^n-th roots ζ of unity with $n \geq 1$. Then it follows that $f(T)$ divides $G(c(1+T)^{-1} - 1)$ by the reflection theorem [4, Proposition 1]. We checked that the power series $g_{\chi}(T)$ for the above χ is not divisible by a distinguished polynomial of degree one in $\boldsymbol{Z}_3[T]$. Therefore, if $\lambda > 0$, we must have $\lambda \geq 2$, and hence $|A_j(k(d_3))| \geq 2j$ for all $j \geq 0$ by Lemma 5. This contradicts the data given in Table 2. $\square$

3.3 The case $p = 2$

In this subsection, we show the following:

Proposition 3 *For $p = 2$, we have $\lambda_3^- = 2$, $\lambda_2^- = \lambda_1^- = \lambda_0^- = 0$ and $\lambda_3^+ = 0$.*

Proof. Let n be an integer with $0 \leq n \leq 3$. For a $\bar{\boldsymbol{Q}}_3$-valued characters χ of G_n^+, we computed the invariant λ_{χ^*} using (9), and obtained the assertion on λ_n^-. It is known that $h_{K_3^+} = 1$ (cf. [20, page 421]). Hence, by using [20, Theorem 10.4], we obtain $\lambda_3^+ = 0$ since 3 does not split in K_3^+. □

Our computation in this section was carried out by using gcc on a PC with Intel Core (TM) i7 CPU 950 @ 3.07 GHz and 8GB memory. It took about one week to compute upper bounds for $B_5(k(d_3))$ and $B_6(k(d_3))$. It took about a few hours to compute the others.

4 Computation of λ_{χ^*}

As in Section 2, let χ be an even Dirichlet character of conductor $p^{n+1+\delta}$ and $\omega = \omega_3$ be the Teichmüller character of conductor 3. Here, $\delta = 1$ or 0 according as $p = 2$ or not. In this section we prove the following proposition and show Theorems 1 and 3.

Proposition 4 (I) *For a prime p with $5 \leq p \leq 599$, we have $\lambda_{\chi^*} = 0$ for all χ and all $n \geq 1$.*

(II) *When $p = 2$ and $n \geq 4$, we have $\lambda_{\chi^*} = 0$ for all χ.*

Proof of Theorems. Theorem 1 follows from Proposition 4(II) and (1), (3). Theorem 2 follows from Theorem 1 and Proposition 2. Theorem 3 follows from Propositions 3 and 4(I) and (1), (3). □

For proving Proposition 4, we make use of Lemma 1. That is, we are going to show that $B_{1,\chi\omega}$ is prime to 3.

First, we introduce some notation. For an integer a and $n \geq 0$, let $s_n(a)$ be the integer satisfying

$$s_n(a) \equiv a \bmod p^{n+1+\delta} \quad \text{and} \quad 0 \leq s_n(a) < p^{n+1+\delta},$$

and define a function $f_n(a)$ $(a \in \boldsymbol{Z})$ by

$$f_n(a) = \begin{cases} 1 & \text{if } p^{n+1+\delta}/3 < s_n(a) < 2p^{n+1+\delta}/3 \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that

$$f_n(-a) = f_n(a) \tag{11}$$

holds for an integer a.

Now we express $B_{1,\chi\omega}$ by using $f_n(a)$.

Lemma 7 *With the notation above, we have*

$$\frac{\omega(p^{n+1+\delta})^{-1}\chi(3)^{-1}}{2} B_{1,\chi\omega} = \frac{1}{2} \sum_{a \bmod p^{n+1+\delta}} f_n(a)\chi(a). \tag{12}$$

Proof. In this proof, we put $P = p^{n+1+\delta}$. For an integer $\hat{a}$, denote by $\hat{s}_n(\hat{a})$ the integer satisfying $\hat{s}_n(\hat{a}) \equiv \hat{a} \bmod 3P$ and $0 \leq \hat{s}_n(\hat{a}) < 3P$. Then, by definition,

$$\frac{1}{2} B_{1,\chi\omega} = \frac{1}{6P} \sum_{\hat{a} \bmod 3P} \hat{s}_n(\hat{a})\chi\omega(\hat{a}). \tag{13}$$

Putting $\hat{a} = 3a + Pb$, where a moves modulo P and $b = 0, 1, 2$, we have

$$\chi\omega(\hat{a}) = \chi(\hat{a})\omega(\hat{a}) = \chi(3a)\omega(Pb) = \omega(P)\chi(3)\omega(b)\chi(a).$$

Hence (13) gives

$$\begin{aligned} \frac{\omega(P)^{-1}\chi(3)^{-1}}{2} B_{1,\chi\omega} &= \frac{1}{6P} \sum_{a \bmod P} \sum_{b=0,1,2} \hat{s}_n(3a + Pb)\omega(b)\chi(a) \\ &= \frac{1}{6P} \sum_{a \bmod P} \left(\hat{s}_n(3a + P) - \hat{s}_n(3a + 2P)\right)\chi(a). \end{aligned}$$

We can directly obtain (12) from this equation, because it is easy to verify

$$\hat{s}_n(3a + P) - \hat{s}_n(3a + 2P) = -P + 3Pf_n(a)$$

(note that $\sum_{a \bmod P} \chi(a) = 0$ holds). $\square$

We first show Proposition 4(II), which is rather short.

Proof of Proposition 4(II). Let $p = 2$ and $n \geq 4$. We can rewrite the right hand side of (12) to obtain

$$\frac{\omega(2^{n+2})^{-1}\chi(3)^{-1}}{2}B_{1,\chi\omega} = \sum_{a \equiv 1 \bmod 4} f_n(a)\chi(a) \tag{14}$$

because every odd integer is congruent modulo 2^{n+2} to $\pm a$ with $a \equiv 1 \bmod 4$ and χ is an even character (also recall that (11) holds). Now, assume that $B_{1,\chi\omega}$ is not prime to 3. Then there is an prime ideal $\tilde{\mathcal{L}}$ of $\boldsymbol{Q}(\zeta_{2^n})$ which lies over 3 and satisfies $B_{1,\chi\omega} \equiv 0 \bmod \tilde{\mathcal{L}}$ (note that $B_{1,\chi\omega}$ lies in $\boldsymbol{Q}(\zeta_{2^n})$ because χ is of order 2^n). Hence, by (14), we have a congruence

$$\sum_{a \equiv 1 \bmod 4} f_n(a)\chi(a) \equiv 0 \bmod \tilde{\mathcal{L}}. \tag{15}$$

Denoting by $\mathcal{L}$ the prime ideal of $\boldsymbol{Q}(\zeta_8)$ lying below $\tilde{\mathcal{L}}$, we know that $\tilde{\mathcal{L}}$ is the only prime ideal over $\mathcal{L}$, i.e. $\mathcal{L}$ remains prime in $\boldsymbol{Q}(\zeta_{2^n})$. So, (15) implies

$$Tr\left(\sum_{a \equiv 1 \bmod 4} f_n(a)\chi(a)\right) \equiv 0 \bmod \mathcal{L}, \tag{16}$$

where Tr denotes the trace map from $\boldsymbol{Q}(\zeta_{2^n})$ to $\boldsymbol{Q}(\zeta_8)$. Further, we know that $Tr(\chi(a)) = 2^{n-3}\chi(a)$ or 0 according as $\chi(a)^8 = 1$ or not. Since $\chi(a)^8 = 1$ if and only if $a \equiv 1 \bmod 2^{n-1}$, we have from (16)

$$\sum_{a \equiv 1 \bmod 2^{n-1}} f_n(a)\chi(a) \equiv 0 \bmod \mathcal{L}. \tag{17}$$

Because integers a modulo 2^{n+2} satisfying $a \equiv 1 \bmod 2^{n-1}$ are represented as $(1 + 2^{n-1})^k$ for $0 \leq k \leq 7$, (17) implies

$$\sum_{k=0}^{7} f_n((1 + 2^{n-1})^k)\zeta^k \equiv 0 \bmod \mathcal{L}, \tag{18}$$

where $\zeta = \chi(1 + 2^{n-1})$ is a primitive 8-th root of unity. Thanks to the assumption $n \geq 4$, we have $(1 + 2^{n-1})^k \equiv 1 + k\, 2^{n-1} \bmod 2^{n+2}$. Hence we see from the definition of the function f_n that $f_n((1 + 2^{n-1})^k) = 1$ if and only if $k = 3, 4, 5$. Therefore, the left hand side of (18) is equal to $\zeta^3 + \zeta^4 + \zeta^5 = -1 - \zeta + \zeta^3$, which is not 0 modulo $\mathcal{L}$ (actually $(-1 - \zeta + \zeta^3)(1 - \zeta + \zeta^3) \equiv 1 \bmod \mathcal{L}$). This contradicts (18). Hence $B_{1,\chi\omega}$ must be prime to 3, which proves Proposition 4(II) by virtue of Lemma 1. $\square$

Hereafter we assume that p is an odd prime, and prove Proposition 4(I). By Lemma 3 and (3), it suffices to prove $\lambda_{\chi^*} = 0$ for $1 \leq n < \boldsymbol{m}_p$.

Let $n_0 = \mathrm{ord}_p(3^{p-1} - 1)$ as in Section 2. Then an easy calculation shows that, for $p \leq 599$, $n_0 = 1$ except for $p = 11$ and when $p = 11$, $n_0 = 2$. In the case $p = 11$ and $n = 1$, it is not hard to compute $B_{1,\chi\omega}$ directly. Concretely, we can compute the norm $N\left(\frac{1}{2}B_{1,\chi\omega}\right)$ of $\frac{1}{2}B_{1,\chi\omega}$ for each character χ. Here, $\boldsymbol{Q}(\chi)$ is the field generated over $\boldsymbol{Q}$ by the values of χ, and N denotes the norm map from $\boldsymbol{Q}(\chi)$ to $\boldsymbol{Q}$. The value of $N\left(\frac{1}{2}B_{1,\chi\omega}\right)$ is $2^{10} \times 89$ and 20891667283264099631 (prime number) when the order of χ is 11 and 55, respectively. Since both values are prime to 3, Lemma 1 implies that $\lambda_{\chi^*} = 0$ for any χ when $p = 11$ and $n = 1$.

So it turns out from Lemma 1 that, for proving Proposition 4(I), it is sufficient to show

$$B_{1,\chi\omega} \text{ is prime to 3 for } 5 \leq p \leq 599 \text{ and } n_0 \leq n < \boldsymbol{m}_p \tag{19}$$

for each character χ.

Here we rewrite the right hand side of (12). Take a primitive root g modulo p^{n+1}, and denote by q the integer which is prime to 3 and satisfies $\dfrac{p-1}{2} = 3^t q$ for some $t \geq 0$. We decompose the character χ as $\chi = \gamma\eta\psi_n$, where ψ_n has conductor p^{n+1} and order p^n, and γ (resp. η) has conductor p and order a power of 3 (resp. a divisor of q), respectively.

Lemma 8 *With the notation above, we have*

$$\begin{aligned}&\frac{\omega(p^{n+1})^{-1}\chi(3)^{-1}\chi(g^{i_0})^{-1}}{2}B_{1,\chi\omega}\\ &= \sum_{0\leq k<qp^n}\ \sum_{l \bmod 3^t} f_n(g^{i_0+3^t k+2qp^n l})\gamma(g^{2qp^n l})\eta(g^{3^t k})\psi_n(g^{3^t k})\end{aligned} \tag{20}$$

for any integer i_0.

Proof. In the right hand side of (12), we can put $a = g^i$, where i moves modulo $(p-1)p^n$ since $\chi(a) = 0$ when a is not prime to p. Hence the left hand side of (20) equals

$$\frac{1}{2}\sum_{i \bmod (p-1)p^n} f_n(g^i)\chi(g^{i-i_0}) = \frac{1}{2}\sum_{i \bmod (p-1)p^n} f_n(g^{i_0+i})\chi(g^i). \tag{21}$$

Here we put $i = 3^t k + 2qp^n l$, where k and l moves modulo $2qp^n$ and 3^t, respectively (note $p-1 = 3^t \times 2q$). Then (21) equals

$$\frac{1}{2} \sum_{k \bmod 2qp^n} \sum_{l \bmod 3^t} f_n(g^{i_0+3^t k+2qp^n l})\gamma(g^{2qp^n l})\eta(g^{3^t k})\psi_n(g^{3^t k}) \tag{22}$$

because $\chi = \gamma\eta\psi_n$ and $\gamma(g^{3^t}) = \eta(g^q) = \psi_n(g^{p^n}) = 1$. In (22), we see that the terms for k and $k + qp^n$ give the same value, because the function f_n is even (see (11)), the characters η, ψ_n are even and $g^{(p-1)p^n/2} \equiv -1 \bmod p^{n+1}$. Therefore, collecting terms for k and $k + qp^n$ in (22), we obtain (20). □

The following Lemma 9 gives a method for verifying (19).

Lemma 9 *Assume $p \geq 5$ and $n \geq n_0$, and denote by $\boldsymbol{F}_3$ the finite field with 3 elements. If there exists an integer i_0 for which the polynomial*

$$\sum_{v=0}^{qp^{n_0}-1} \left(\sum_{l=0}^{3^t-1} f_n(g^{i_0+3^t p^{n-n_0} v+2qp^n l}) \right) x^v \bmod 3 \tag{23}$$

in $\boldsymbol{F}_3[x]$ is prime to $(x^{qp^{n_0}} - 1)/(x^{qp^{n_0-1}} - 1) \bmod 3$, then $B_{1,\chi\omega}$ is prime to 3 for any even character χ of conductor p^{n+1}.

Proof. Assume that $B_{1,\chi\omega}$ is not prime to 3. This assumption implies that there exists a prime ideal $\tilde{\mathcal{L}}$ of $\boldsymbol{Q}(\zeta_{(p-1)p^n})$ lying over 3 which divides $B_{1,\chi\omega}$. Then, by Lemma 8, we have a congruence

$$\sum_{0 \leq k < qp^n} \sum_{l \bmod 3^t} f_n(g^{i_0+3^t k+2qp^n l})\gamma(g^{2qp^n l})\eta(g^{3^t k})\psi_n(g^{3^t k}) \equiv 0 \bmod \tilde{\mathcal{L}} \tag{24}$$

for any integer i_0. Since $\gamma(g^{2qp^n l})$ is a root of unity of 3-power order and $\tilde{\mathcal{L}}$ lies over 3, we have $\gamma(g^{2qp^n l}) \equiv 1 \bmod \tilde{\mathcal{L}}$ for any l. Hence, determining integers $\tilde{a}_k$ by

$$\tilde{a}_k = \sum_{l \bmod 3^t} f_n(g^{i_0+3^t k+2qp^n l}),$$

we obtain from (24)

$$\sum_{0 \leq k < qp^n} \tilde{a}_k \, \eta(g^{3^t k})\psi_n(g^{3^t k}) \equiv 0 \bmod \tilde{\mathcal{L}}. \tag{25}$$

Here we note that the left hand side of (25) belongs to the field $\boldsymbol{Q}(\zeta_{qp^n})$. Let $\mathcal{L}$ be the prime ideal of $\boldsymbol{Q}(\zeta_{qp^{n_0}})$ lying below $\tilde{\mathcal{L}}$. Then, by the definition of n_0,

$\tilde{\mathcal{L}}$ is the only prime ideal lying over $\mathcal{L}$, i.e. $\mathcal{L}$ does not decompose in $\boldsymbol{Q}(\zeta_{qp^n})$. Hence we can conclude from (25) that

$$Tr\left(\sum_{0\le k<qp^n} \tilde{a}_k\, \eta(g^{3^t k})\psi_n(g^{3^t k})\right) \equiv 0 \bmod \mathcal{L} \tag{26}$$

holds, where Tr denotes the trace map from $\boldsymbol{Q}(\zeta_{qp^n})$ to $\boldsymbol{Q}(\zeta_{qp^{n_0}})$. Since $\psi_n(g^{3^t})$ is a primitive p^n-th root of unity, $Tr(\psi_n(g^{3^t k})) = 0$ unless k is divisible by p^{n-n_0}. Therefore, putting $k = p^{n-n_0}v$, we obtain from (26)

$$\sum_{0\le v<qp^{n_0}} a_v\, \eta(g^{3^t p^{n-n_0} v})\psi_n(g^{3^t p^{n-n_0} v}) \equiv 0 \bmod \mathcal{L}, \tag{27}$$

where $a_v = \tilde{a}_{p^{n-n_0}v}$. We see that $\xi = \eta(g^{3^t p^{n-n_0}})\psi_n(g^{3^t p^{n-n_0}})$ is a primitive dp^{n_0}-th root of unity for some divisor d of q, because $\eta^q = 1$ and the order of ψ_n is p^n. This means that ξ is a root of the polynomial $(x^{qp^{n_0}}-1)/(x^{qp^{n_0-1}}-1)$. Therefore, validity of the congruence (27) implies that the two integer-coefficient polynomials $\sum_{0\le v<qp^{n_0}} a_v x^v$ and $(x^{qp^{n_0}}-1)/(x^{qp^{n_0-1}}-1)$ have a non-trivial common factor when they are reduced modulo 3. This contradicts the assumption of Lemma 9. Hence $B_{1,\chi\omega}$ must be prime to 3, which finishes the proof of Lemma 9. □

We verified Proposition 4(I) by using Lemma 9, Lemma 1 and a computer. Note that we took the integer i_0 in Lemma 9 as a multiple of $p-1$, i.e. we put $i_0 = (p-1)j$ for an integer j. For given p and n, we first take $j = 0$ and checked if the polynomial (23) is prime to $(x^{qp^{n_0}}-1)/(x^{qp^{n_0-1}}-1)$ mod 3. If they are prime to each other, we are done. If not, we take $j = 1$ and compute (23) $\cdots$. Repeating this process for $j = 1, 2, \cdots$, we could find an appropriate j for every p and n in (19), proving Proposition 4(II). Our computation was carried out by using Maple 15 (cf. [15]) on Apple's Mac Pro computer with two 2.4 GHz quad-core Intel Xeon processor and 16GB memory. It took about a week to carry out the whole calculation.

In the above process, long time was needed for computation when p is large and 3 is not a primitive root for p. For reference, we show a list of p with large values of $\boldsymbol{m}_p$, i.e. $\boldsymbol{m}_p \ge 400$ (Table 3).

It is notable that the first value $j = 0$ was valid for our purpose in most cases. We list in Table 4 the positive values of j that was needed to conclude that $B_{1,\chi\omega}$ is prime to 3 in the range $p \le 599$.

Table 1: Iwasawa invariants $\lambda_0^- \geq 1$

p	$3p$	d_χ	d'_χ	λ_{χ^*}	λ_0^-	p	$3p$	d_χ	d'_χ	λ_{χ^*}	λ_0^-
19	57	3	2	2		337	1011	2	1	2	
		9	6	2	16			6	2	2	6
29	87	2	1	1	1	353	1059	2	1	1	
37	111	3	2	2				4	2	1	3
		9	6	2	16	379	1137	3	2	8	
73	219	3	2	2				9	6	8	
		9	6	2	16			27	18	8	208
109	327	2	1	1		397	1191	2	1	1	
		3	2	8				3	2	2	
		6	2	1				6	2	1	
		9	6	8				9	6	2	
		18	6	1				18	6	1	25
		27	18	8		401	1203	2	1	3	
		54	18	1	235			8	2	1	5
113	339	2	1	1	1	433	1299	3	2	8	
127	381	3	2	2				9	6	8	
		9	6	2	16			27	18	8	208
137	411	2	1	1	1	443	1329	13	3	1	3
163	489	3	2	26		449	1347	2	1	1	1
		9	6	26		457	1371	2	1	1	
		27	18	26				6	2	1	3
		81	54	26	2080	461	1383	2	1	1	1
173	519	2	1	1	1	487	1461	3	2	80	
181	543	2	1	2				9	6	80	
		3	2	2				27	18	80	
		6	2	2				81	54	80	
		9	6	2				243	162	80	19360
		18	6	2	34	521	1563	2	1	1	
199	597	3	2	2				26	3	1	4
		9	6	2	16	523	1569	3	2	2	
229	687	2	1	1				9	6	2	16
		6	2	1	3	541	1623	3	2	8	
257	771	2	1	1	1			9	6	8	
271	813	3	2	8				27	18	8	208
		9	6	8		577	1731	3	2	2	
		27	18	8	208			8	2	1	
281	843	20	4	1	4			9	6	2	
307	921	3	2	2				24	6	1	
		9	6	2	16			72	18	1	42
313	939	26	3	1		599	1797	13	3	1	3
		78	9	1	12						

Table 2: $\lambda_0^- \geq 1$ and $\lambda_0^+ = 0$

p	$3p$	d_1	d_2	$B'_0(B_0)$	B'_1	B'_2	d_3	B'_0	B'_1	B'_2	B'_3	B'_4	B'_5	B'_6
19	57	9	1	(1)			1	(1)	(-)					
29	87	2	1	(1)			1	(1)	(-)					
37	111	9	1	(1)			1	(1)	(-)					
73	219	9	1	(1)			9	(1)	(3)	(9)	(-)			
109	327	54	1	(1)			54	(1)	(3)	(9)	(27)	(81)	(-)	
113	339	2	1	(1)			1	(1)	(-)					
127	381	9	1	(1)			1	(1)	(-)					
137	411	2	1	(1)			1	(1)	(-)					
163	489	81	1	(1)			1	(1)	(-)					
173	519	2	1	(1)			1	(1)	(-)					
181	543	18	1	(1)			18	(1)	(3)	(9)	(27)	(-)		
199	597	9	1	(1)			1	(1)	(-)					
229	687	6	6	(3)	(9)	(-)	6	(3)	(9)					
257	771	2	2	(3)	(-)		1	(1)	(-)					
271	813	27	1	(1)			27	(1)	(3,3)	(9,9)	(27,27)	(-)		
281	843	20	1	(1)			1	(1)	(-)					
307	921	9	1	(1)			9	(1)	(3)	(9)				
313	939	78	1	(1)			6	(1)	(-)					
337	1011	6	1	(1)			6	(1)	(3)	(9)	(-)			
353	1059	4	1	(1)			1	(1)	(-)					
379	1137	27	1	(1)			1	(1)	(-)					
397	1191	18	1	(1)			18	(1)	(3)	(9)	(27)	(81)	(243)	(-)
401	1203	8	8	(3,3)	(-)		1	(1)	(-)					
433	1299	27	1	(1)			1	(1)	(-)					
443	1329	13	1	(1)			1	(1)	(-)					
449	1347	2	1	(1)			1	(1)	(-)					
457	1371	6	1	(1)			6	(1)	(3)	(9)	(-)			
461	1383	2	1	(1)			1	(1)	(-)					
487	1461	243	1	(1)			1	(1)	(-)					
521	1563	26	26	(3,3,3)	(-)		1	(1)	(-)					
523	1569	9	1	(1)			9	(1)	(3,3,3)	(3,9,9)	(-)			
541	1623	27	1	(1)			1	(1)	(-)					
577	1731	72	1	(1)			36	(1)	(3)	(9)	(-)			
599	1797	13	1	(1)			1	(1)	(-)					

Table 3: Large values of $\boldsymbol{m}_p$

p	383	443	467	479	503	563	577	587	599
$\boldsymbol{m}_p$	403	406	491	503	528	591	405	616	557

Table 4: Positive values of j

p	11	13		
$\boldsymbol{m}_p$	14	9		
n	4	2	4	6
j	1	1	1	2

Corrigendum In the proof of [8, Theorem 3], we have written that "$\cdots$ the unit index of $L_{n,j}$ equals 1 by Conner and Hurrelbrink [1, (13.4), (13.5)]." However, this is slightly incorrect. We have to change "$\cdots$ by Conner and Hurrelbrink $\cdots$" to "$\cdots$ by Satz 22 of Hasse's monograph *Über die Klassenzahl abelscher Zahlkörper*."

References

[1] E. Friedman, Ideal class groups in basic $\boldsymbol{Z}_{p_1} \times \cdots \times \boldsymbol{Z}_{p_s}$-extensions of abelian number fields, Invent. Math., **65** (1982), 425-440.

[2] T. Fukuda, Remarks on $\boldsymbol{Z}_p$-extensions of number fields, Proc. Japan Acad., **70** (1994), 264-266.

[3] R. Greenberg, On the Iwasawa invariants of totally real number fields, Amer. J. Math., **98** (1976), 263-284.

[4] R. Greenberg, On p-adic L-functions and cyclotomic fields II, Nagoya Math. J., **67** (1977), 139-158.

[5] K. Horie, The ideal class group of the basic $\boldsymbol{Z}_p$-extension over an imaginary quadratic field, Tohoku Math. J., **57** (2005), 375-394.

[6] K. Horie, Triviality in ideal class groups of Iwasawa-theoretical abelian number fields, J. Math. Soc Japan, **57** (2005), 827-857.

[7] H. Ichimura, On the parity of the class number of an imaginary abelian field of conductor $2^a p^b$, Arch. Math. (Basel), **96** (2011), 555-563.

[8] H. Ichimura, On the class group of a cyclotomic $\boldsymbol{Z}_p \times \boldsymbol{Z}_\ell$-extension, Acta Arith., in press.

[9] H. Ichimura and S. Nakajima, On the 2-part of the class numbers of cyclotomic fields of prime power conductors, J. Math. Soc. Japan, in press.

[10] H. Ichimura and H. Sumida, On the Iwasawa invariants of certain real abelian fields II, Internat. J. Math., **7** (1996), 721-744.

[11] K. Iwasawa, On $\boldsymbol{Z}_\ell$-extensions of algebraic number fields, Ann. of Math., **98** (1973), 246-326.

[12] Y. Koyama and K. Yoshino, Prime divisors of the class number of the real p^r-th cyclotomic field and characteristic polynomials attached to them, RIMS Kôkyûroku Bessatsu, **B12** (2009), 149-172.

[13] J. Kraft and R. Schoof, Computing Iwasawa modules of real quadratic number fields, Compositio Math., **97** (1995), 135-155.

[14] S. Lang, Cyclotomic fields I and II. (Combined second edition), Springer-Verlag, New York, 1990.

[15] Maplesoft, http://www.maplesoft.com/products/maple/index.aspx.

[16] B. Mazur and A. Wiles, Class fields of abelian extensions over $\boldsymbol{Q}$, Invent. Math., **76** (1984), 179-330.

[17] W. Sinnott, On the Stickelberger ideal and the circular units of an abelian field, Invent. Math., **62** (1980), 181-234.

[18] H. Sumida, Greenberg's conjecture and the Iwasawa polynomial, J. Math. Soc. Japan, **49** (1997), 689-711.

[19] H. Sumida, On capitulation of S-ideals in $\boldsymbol{Z}_p$-extensions, J. Number Theory, **86** (2001), 163-174.

[20] L. C. Washington, Introduction to Cyclotomic Fields (2nd ed.), Springer, New York, 1997.

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