On the 2-adic Iwasawa lambda invariants of the p-cyclotomic fields and their quadratic twists

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Abstract

Let p be an odd prime number, $K_n = \boldsymbol{Q}(\zeta_{p^{n+1}})$ the p^{n+1}st cyclotomic field and h_n^- the relative class number of K_n. Fixing a negative integer $d \in \boldsymbol{Z}$ with $\sqrt{d} \notin K_0$, we put $L_n = K_n^+(\sqrt{d})$. Denote by λ_n^- and λ_n^* the minus parts of the 2-adic lambda invariants of K_n and L_n, respectively. By a theorem of Friedman, these invariants are stable for sufficiently large n. First, under the assumption that h_n^-/h_{n-1}^- is odd for all $n \geq 1$, we give a quite explicit version of this result. Second, we show the assumption is satisfied for all $p \leq 599$. Further, using these results, we compute the invariants λ_n^- and λ_n^* with $d = -1, -3$ for all $p \leq 599$ and all n with the help of computer.

1 Introduction

We fix an odd prime number p, and let $K_n = \boldsymbol{Q}(\zeta_{p^{n+1}})$ $(n \geq 0)$ denote the p^{n+1}st cyclotomic field, where for an integer $m \geq 2$, ζ_m is a primitive mth root of unity. We put $K_\infty = \cup_n K_n$. We also fix an integer $d \in \boldsymbol{Z}$ with $\sqrt{d} \notin K_0$. We denote by $L_n = L_{d,n}$ the imaginary quadratic subextension of the imaginary (2, 2)-extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$, where N^+ is the maximal real subfield of an imaginary abelian field N. Then $L_\infty = L_{d,\infty} = \cup_n L_n$ is the cyclotomic $\boldsymbol{Z}_p$-extension of L_0. Here, for a prime number ℓ, $\boldsymbol{Z}_\ell$ denotes the ring of ℓ-adic integers. In [7], we called L_n and the $\boldsymbol{Z}_p$-extension L_∞/L_0 the quadratic twists of K_n and K_∞/K_0 with respect to the integer d, respectively. Let S_d be the set of prime numbers $\ell \neq p$ ramified at the quadratic extension $\boldsymbol{Q}(\sqrt{d})/\boldsymbol{Q}$, which is nonempty as $\sqrt{d} \notin K_0$. We put

$$n_d = \max\left(\mathrm{ord}_p(\ell^{p-1}-1) \mid \ell \in S_d\right).$$

Here, $\mathrm{ord}_p(*)$ denotes the additive valuation at p normalized with $\mathrm{ord}_p(p) = 1$. Let h_n^- and h_n^* be the relative class numbers of K_n and L_n, respectively. The ratios h_n^-/h_{n-1}^- and h_n^*/h_{n-1}^* are odd for sufficiently large n by a well known theorem of Washington [13] on the non-p-part of the class number in a cyclotomic $\boldsymbol{Z}_p$-extension. In [7], we obtained the following result on the parity of these ratios.

Theorem 1 *Under the above setting, assume that the ratio h_n^-/h_{n-1}^- is odd for all $n \geq 1$. Then, h_n^*/h_{n-1}^* is odd for all $n \geq n_d$.*

For a number field N, we denote by λ_N the Iwasawa lambda invariant of the ideal class group of the cyclotomic $\boldsymbol{Z}_2$-extension $N^{(2)}/N$. When N is an imaginary abelian field, we put $\lambda_N^- = \lambda_N - \lambda_{N^+}$ (≥ 0). We write $\lambda_n^- = \lambda_{K_n}^-$ and $\lambda_n^* = \lambda_{L_n}^-$ for brevity. We have $\lambda_{n+1}^- \geq \lambda_n^-$ and $\lambda_{n+1}^* \geq \lambda_n^*$ for all n. The invariants λ_n^- and λ_n^* are stable for sufficiently large n by a theorem of Friedman [2] which was obtained by sharpening the arguments in [13]. In this paper, using Theorem 1, we give the following explicit version of Friedman's theorem. We put

$$\tilde{n}_d = \max\left(n_d,\, \mathrm{ord}_p(2^{p-1}-1)\right) \quad \text{and} \quad \tilde{n}_1 = \mathrm{ord}_p(2^{p-1}-1).$$

Theorem 2 *Under the above setting, assume that the ratio h_n^-/h_{n-1}^- is odd for all $n \geq 1$. Then, $\lambda_n^- = \lambda_{n-1}^-$ for all $n \geq \tilde{n}_1$, and $\lambda_n^* = \lambda_{n-1}^*$ for all $n \geq \tilde{n}_d$.*

In the previous paper [8], we have shown that the assumption is satisfied for $p \leq 509$. We extend it as follows.

Theorem 3 *When $p \leq 599$, the ratio h_n^-/h_{n-1}^- is odd for all $n \geq 1$.*

From Theorems 2 and 3, we immediately obtain the following quite explicit version of Friedman's theorem.

Theorem 4 *When $p \leq 599$, we have $\lambda_n^- = \lambda_{n-1}^-$ for all $n \geq \tilde{n}_1$, and $\lambda_n^* = \lambda_{n-1}^*$ for all $n \geq \tilde{n}_d$.*

Using this theorem, we calculate in Section 4 all the invariants λ_n^- and λ_n^* for $d = -1, -3$ when $p \leq 599$.

Remarks 1. (I) When $d = -1$, the assertion on λ_n^* in Theorem 2 is contained in [5].

(II) Another type of explicit version of Friedman's theorem is given in [6]. Let $F_n = K_n^+(\zeta_3)$, and λ_n be the Iwasawa lambda invariant of the cyclotomic $\boldsymbol{Z}_3$-extension over F_n. In [9], using results in [6], we calculated λ_n for all $p \leq 599$ and all n.

(III) Let $p = 3$. Let L be an imaginary abelian field, and L_n the nth layer of the cyclotomic $\boldsymbol{Z}_3$-extension over L. In this special setting, Friedman and Sands [3] obtained an explicit version of Friedman's theorem for the minus part of the ℓ-adic lambda invariant of L_n. Here, ℓ is a prime number with $\ell \geq 5$.

2 Proof of Theorem 2

For an integer $j \geq 0$, let $K_{n,j}$ (resp. $L_{n,j}$) be the jth layer of the cyclotomic $\boldsymbol{Z}_2$-extension $K_n^{(2)}/K_n$ (resp. $L_n^{(2)}/L_n$). To deal with the invariants λ_n^- and λ_n^* simultaneously, we denote the relative class numbers of $K_{n,j}$ and $L_{n,j}$ by the same symbol $h_{n,j}^-$. When the base field is K_n (resp. L_n), $h_{n,0}^-$ equals h_n^- (resp. h_n^*). We see that the equality $\lambda_n^- = \lambda_{n-1}^-$ (resp. $\lambda_n^* = \lambda_{n-1}^*$) holds if and only if the ratio $h_{n,j}^-/h_{n-1,j}^-$ is odd for all sufficiently large j. Therefore, to prove Theorem 2, it suffices to show the following proposition by virtue of Theorem 1.

Proposition 1 *Let n be an integer with $n \geq \mathrm{ord}_p(2^{p-1} - 1)$. If the ratio $h_{n,0}^-/h_{n-1,0}^-$ is odd, then $h_{n,j}^-/h_{n-1,j}^-$ is odd for all $j \geq 1$.*

It is well known that the unit index of K_n equals 1 (see [14, Corollary 4.13]). Further, it is known that the unit indexes of $L_{n,j}$ and $L_{n-1,j}$ coincide (see [7, Lemma 4]). Let $p^* = p$ (resp. 1) when the base field equals K_n (resp. L_n). Then it follows from the class number formula [14, Theorem 4.17] that

$$h_{n,j}^-/h_{n-1,j}^- = p^* \cdot \prod_{\eta,\psi,\theta} \left(-\frac{1}{2} B_{1,\eta\psi\theta} \right), \tag{1}$$

where η runs over the odd Dirichlet characters associated to K_0 or L_0, and ψ (resp. θ) runs over the even Dirichlet characters of conductor p^{n+1} and order p^n (resp. conductor dividing 2^{j+2}). We regard these characters as $\bar{\boldsymbol{Q}}_2$-valued where $\bar{\boldsymbol{Q}}_2$ is an algebraic closure of the 2-adic rationals $\boldsymbol{Q}_2$. Further, for an odd Dirichlet character χ with conductor $f = f_\chi$, $B_{1,\chi}$ denotes the generalized Bernoulli number:

$$B_{1,\chi} = \frac{1}{f}\sum_{a=1}^{f-1} a\chi(a).$$

To prove Proposition 1, let us recall some properties of 2-adic L-functions. Denote by ω_4 the odd character of conductor 4. Let χ be an odd Dirichlet character of the 1st kind at the prime 2 with $\chi \neq \omega_4$, and $\mathcal{O}_\chi = \boldsymbol{Z}_2[\chi]$ the subring of $\bar{\boldsymbol{Q}}_2$ generated over $\boldsymbol{Z}_2$ by the values of χ. Let c be the odd part of the conductor of the even nontrivial character $\omega_4\chi$. Iwasawa constructed a power series $H_\chi(T) \in \mathcal{O}_\chi[[T]]$ related to the 2-adic L-function $L_2(s,\, \omega_4\chi)$ by

$$H_\chi((1+4c)^s - 1) = \frac{1}{2}L_2(s,\, \omega_4\chi) \tag{2}$$

for $s \in \boldsymbol{Z}_2$. (Under the notation of [14, §7.2], H_χ is written as $f(T,\, \omega_4\chi)$.) Further, for an even character θ of 2-power conductor, the power series $H_\chi(T)$ enjoys the following property:

$$H_\chi(\zeta_\theta(1+4c)^s - 1) = \frac{1}{2}L_2(s,\, \omega_4\chi\theta) \tag{3}$$

for $s \in \boldsymbol{Z}_2$, where ζ_θ is a 2-power-th root of unity associated to θ. For (2) and (3), see [14, Theorem 7.10]. Further, as is well known,

$$L_2(0,\, \omega_4\chi) = -(1-\chi(2))B_{1,\chi} \tag{4}$$

holds (see [14, Theorem 5.11]).

Proof of Proposition 1. Let ν_8 be the unique even character of conductor 8. We put $\tilde{\eta} = \eta\nu_8^{-1}$ or η according as the conductor f_η of η is divisible by 8 or not. Then we see that the odd characters $\tilde{\eta}$ and $\tilde{\eta}\psi$ are of the 1st kind at the prime 2. Assume that $h^-_{n,0}/h^-_{n-1,0}$ is odd. Then, by (1), we have

$$\left(\frac{1}{2}B_{1,\eta\psi},\, 2\right) = 1. \tag{5}$$

First we deal with the case $8 \nmid f_\eta$. Then $\tilde{\eta} = \eta$ by definition. By (2) and (4), we have

$$H_{\eta\psi}(0) = -\frac{1}{2}(1-\eta\psi(2))B_{1,\eta\psi}. \tag{6}$$

As the order of η divides $p-1$ and that of ψ is p^n, the value $\eta\psi(2)$ is a 2-power-th root of unity only when $\psi(2)=1$. Since the conductor of ψ is p^{n+1}, the last condition holds if and only if $2^{p-1}\equiv 1 \bmod p^{n+1}$, which is impossible because of the assumption $n \geq \mathrm{ord}_p(2^{p-1}-1)$. Therefore, $\eta\psi(2)$ is not a 2-power-th root of unity, and thus $1-\eta\psi(2)$ is a 2-adic unit. It follows from (5) and (6) that $H_{\eta\psi}(0)\in\mathcal{O}_{\eta\psi}^{\times}$, and hence $H_{\eta\psi}(T)$ is a unit of the power series ring $\mathcal{O}_{\eta\psi}[[T]]$. Therefore, for each nontrivial even character θ of 2-power conductor, we see from (3) and (4) that

$$\frac{1}{2}B_{1,\eta\psi\theta}=\frac{1}{2}(1-\eta\psi\theta(2))B_{1,\eta\psi\theta}=-\frac{1}{2}L_2(0,\,\omega_4\eta\psi\theta)=-H_{\eta\psi}(\zeta_\theta-1)$$

is a 2-adic unit. (Here, the first equality holds as the conductor of $\eta\psi\theta$ is even.) Hence, it follows from the class number formula (1) that the ratio $h_{n,j}^-/h_{n-1,j}^-$ is odd for all $j\geq 1$.

Next, we deal with the case $8|f_\eta$. Noting that $\tilde{\eta}=\eta\nu_8^{-1}$, we see from (3) and (4) that

$$H_{\tilde{\eta}\psi}(\zeta_{\nu_8}-1)=\frac{1}{2}L_2(0,\,\omega_4\tilde{\eta}\psi\nu_8)=-\frac{1}{2}(1-\eta\psi(2))B_{1,\eta\psi}.$$

As we have seen above, $1-\eta\psi(2)$ is a 2-adic unit. Hence, it follows from (5) that $H_{\tilde{\eta}\psi}(T)$ is a unit of $\mathcal{O}_{\tilde{\eta}\psi}[[T]]$. Therefore, for each even character θ of 2-power conductor, we observe from (3) and (4) that

$$\begin{aligned}&\frac{1}{2}(1-\eta\psi\theta(2))B_{1,\eta\psi\theta}\\ &\quad=\frac{1}{2}(1-\tilde{\eta}\psi\nu_8\theta(2))B_{1,\tilde{\eta}\psi\nu_8\theta}=-\frac{1}{2}L_2(0,\,\omega\tilde{\eta}\psi\nu_8\theta)=-H_{\tilde{\eta}\psi}(\zeta_{\nu_8\theta}-1)\end{aligned}$$

is a 2-adic unit. The desired assertion follows from this and the formula (1). □

3 Proof of Theorem 3

Theorem 3 extends the computation of [8] which showed that h_n^-/h_{n-1}^- is odd for $p\leq 509$. Proof of Theorem 3 follows one of the methods given in [8, §4] with some improvements on computation. In this section, we first prove Theorem 3, explaining our improvements, and then present our data of computation. In the following, we assume $p \leq 599$, and n denotes a positive integer.

Let $\mathrm{Tr} : \boldsymbol{Q}_2(\zeta_p) \to \boldsymbol{Q}_2$ be the trace map and let $\boldsymbol{a}_p$ be the number of pth roots ζ of unity satisfying $\mathrm{Tr}(\zeta) \equiv 0 \bmod 2$ and $\boldsymbol{b}_p = p - \boldsymbol{a}_p$. (Note that we have $\boldsymbol{a}_p = 1$ when 2 is a primitive root modulo p.) Put

$$\boldsymbol{m}_p = \left[\frac{\phi(p-1)\log(2(p-1)\min(\boldsymbol{a}_p, \boldsymbol{b}_p) - 1)}{\log p} \right],$$

where ϕ is the Euler function and $[x]$ denotes the largest integer $\leq x$. Then, by virtue of [8, Theorem 1], h_n^-/h_{n-1}^- is odd for $n > \boldsymbol{m}_p$.

For verifying that h_n^-/h_{n-1}^- is odd for $1 \leq n \leq \boldsymbol{m}_p$, we define a polynomial $F_{i_0}(T) \in \boldsymbol{F}_2[T]$ in the following way ($\boldsymbol{F}_2 = \boldsymbol{Z}/2\boldsymbol{Z}$ is the field with two elements). We take a primitive root g modulo p^2, factor $p-1$ as

$$p - 1 = 2^t q \quad (t \geq 1,\ q : \text{odd}),$$

and, for any integer a, denote by $s_n(a)$ the integer satisfying

$$s_n(a) \equiv a \bmod p^{n+1} \quad \text{and} \quad 0 \leq s_n(a) < p^{n+1}.$$

Then, for an integer i_0, we put

$$F_{i_0}(T) = \sum_{k=0}^{qp-1} \left(\sum_{l=0}^{2^{t-1}-1} s_n(g^{i_0 + 2^t p^{n-1} k + qp^n l}) \right) T^k \mod 2 \ \in \boldsymbol{F}_2[T],$$

where "mod 2" indicates the reduction modulo 2 of each coefficient of the polynomial.

Our method of proving Theorem 3 relies on the following assertion.

Proposition 2 *We put*

$$\Phi(T) = \frac{T^{qp} - 1}{T^q - 1} \mod 2 \ \in \boldsymbol{F}_2[T].$$

If there exists an integer i_0 for which $F_{i_0}(T)$ is prime to $\Phi(T)$ in $\boldsymbol{F}_2[T]$, then h_n^-/h_{n-1}^- is odd.

Proof. The class number formula (see [14, Theorem 4.17]) gives

$$h_n^-/h_{n-1}^- = p \prod_{\chi} \left(-\frac{1}{2} B_{1,\chi} \right),$$

where χ runs over all odd characters of conductor p^{n+1}. Now assume that h_n^-/h_{n-1}^- is even. Then, there exist a character χ and a prime ideal $\hat{\mathcal{L}}$ of $\boldsymbol{Q}(\zeta_{(p-1)p^n})$ over 2 which satisfy

$$\frac{1}{2}B_{1,\chi} \equiv 0 \mod \hat{\mathcal{L}}.$$

Therefore, for any integer i_0, a congruence

$$\frac{1}{2}\chi(g^{i_0})^{-1}B_{1,\chi} \equiv 0 \mod \hat{\mathcal{L}} \tag{7}$$

holds. Take a prime ideal $\tilde{\mathcal{L}}$ of $\boldsymbol{Q}(\zeta_{(p-1)p})$ lying below $\hat{\mathcal{L}}$. Then $\tilde{\mathcal{L}}$ is inert in the extension $\boldsymbol{Q}(\zeta_{(p-1)p^n})/\boldsymbol{Q}(\zeta_{(p-1)p})$ because $\mathrm{ord}_p(2^{p-1}-1)=1$ for $p \leq 599$. So, by applying the trace map $\mathrm{Tr} : \boldsymbol{Q}(\zeta_{(p-1)p^n}) \to \boldsymbol{Q}(\zeta_{(p-1)p})$ to (7), we obtain

$$\mathrm{Tr}\left(\frac{1}{2}\chi(g^{i_0})^{-1}B_{1,\chi}\right) \equiv 0 \mod \tilde{\mathcal{L}}. \tag{8}$$

The character χ can be expressed in the form $\chi = \delta\varphi\psi$ with characters δ, φ, ψ satisfying the conditions

character	δ	φ	ψ
conductor	p	p	p^{n+1}
order	2^t	a divisor of q	p^n
parity	odd	even	even

(cf. [8, p.334]). Then, by carrying out the computation given in the proof of [8, Lemma 12(I)] in the case $d = q, d' = 1$, we obtain

$$\mathrm{Tr}\left(\frac{1}{2}\chi(g^{i_0})^{-1}B_{1,\chi}\right) \equiv F_{i_0}(\rho) \mod \tilde{\mathcal{L}}, \tag{9}$$

where $\rho = \varphi(g^{2^t})\psi(g^{2^t p^{n-1}})$. Because of the properties of φ and ψ, ρ is a qpth root of unity with $\rho^q \neq 1$. So, ρ satisfies $\Phi(\rho) \equiv 0 \mod \tilde{\mathcal{L}}$. Combining this congruence with (8) and (9), we see that, when reduced modulo $\tilde{\mathcal{L}}$, ρ is a common root of the polynomials $\Phi(T)$ and $F_{i_0}(T)$. This contradicts the assumption that they are prime to each other. Hence, h_n^-/h_{n-1}^- must be odd. □

Proposition 2 gives our first improvement on Method 1 of [8, §4]. That is, by virtue of Proposition 2, all characters φ are settled with one polynomial $F_{i_0}(T)$, while, in [8], we computed a polynomial for each divisor d of q.

For determining the polynomial $F_{i_0}(T)$, we need to compute integers $s_n(g^{i_0+2^t p^{n-1}k+qp^n l})$, which is equal to $s_n(g^{i_0} \cdot s_n(g^{2^t p^{n-1}k+qp^n l}))$. Lemma 1 below gives a way of computing $s_n(g^{2^t p^{n-1}k+qp^n l})$ inductively, which contributes to diminishing computation time. This is our second improvement on the method in [8, §4].

Lemma 1 *Putting* $c_n(k,l) = s_n(g^{2^t p^{n-1}k+qp^n l})$ *for integers* k, l*, we have* $c_{n+1}(k,l) = s_{n+1}(c_n(k,l)^p)$ *for all* $n \geq 1$.

Proof. First, we show that

$$s_{n+1}(a^p) = s_{n+1}(s_n(a)^p)$$

holds for any integer a. Raising the equation $a = s_n(a) + p^{n+1}N$ $(N \in \boldsymbol{Z})$ to the pth power, we have $a^p = (s_n(a) + p^{n+1}N)^p = s_n(a)^p + p^{n+2}M$ for some $M \in \boldsymbol{Z}$. This proves the equation.

Putting $a(k,l) = g^{2^t k+qpl}$, we have $c_n(k,l) = s_n(a(k,l)^{p^{n-1}})$. Then the property given above shows

$$c_{n+1}(k,l) = s_{n+1}\left(a(k,l)^{p^n}\right) = s_{n+1}\left(\left(a(k,l)^{p^{n-1}}\right)^p\right) = s_{n+1}(c_n(k,l)^p),$$

which proves Lemma 1. □

With the aid of Lemma 1 we applied Proposition 2 to primes $p \leq 599$ and integers n in the range $1 \leq n \leq \boldsymbol{m}_p$. As a result of computer calculation, we could find a value of i_0 in all cases, proving Theorem 3. We reported in [8] that in almost all cases in the range $p \leq 509$ and $1 \leq n \leq \boldsymbol{m}_p$, the value $i_0 = 0$ worked for verification (note that $i_0 = 0$ corresponds to $r = 0$ in [8]). The situation turned out to be the same for the new primes $509 < p \leq 599$, that is, the value $i_0 = 0$ worked for all p in this range. In Table 1, we give all p and n for which we must take positive i_0 for applying Proposition 2.

Table 1: positive value of i_0

p	7	31	89	127								
$\boldsymbol{m}_p$	3	15	77	71								
n	2	8	77	12	23	25	26	43	45	48	63	66
i_0	1	1	1	1	1	1	1	1	1	1	3	1

Correction: In Table 3 of [8], a column with the values $p = 89, \boldsymbol{m}_p = n = 77, d = 1, r = 1$ was missing. We apologize for this mistake, and request readers to refer to the correct data in Table 1.

All primes in the range $509 < p \leq 599$ are given in Table 2 with the value $\boldsymbol{m}_p$. (The symbol ◯ in the last row of Table 2 indicates that 2 is a primitive root modulo p^2.) For every p in Table 2 and every n with $1 \leq n \leq \boldsymbol{m}_p$, we could take $i_0 = 0$ in applying Proposition 2.

Table 2: Data for $509 < p \leq 599$

p	521	523	541	547	557	563	569	571	577	587	593	599
$\boldsymbol{m}_p$	383	186	159	159	306	310	559	267	362	323	544	527
p.r.	-	◯	◯	◯	◯	◯	-	-	-	◯	-	-

Our computation was carried out with Maple 16 (cf. [11]) on Apple's iMac computer with 3.4 GHz Intel Core i7 CPU and 16 GB memory. We treated all primes $p \leq 599$, including the primes with $p \leq 509$ which were settled in [8]. The total time of computation was about 40 hours.

4 Computation of λ_0^- and λ_0^*

Let p be an odd prime number with $p \leq 599$. To know all the invariants λ_n^- and λ_n^* for $d = -1, -3$, Theorem 4 tells us that it suffices to compute (i) λ_0^- and λ_0^* except for the case $d = -3$ and $p = 11$ and (ii) λ_0^* and λ_1^* for that case. This is because, as is well known, (a) $\mathrm{ord}_p(2^{p-1} - 1) = 1$ and (b) $\mathrm{ord}_p(3^{p-1} - 1) = 1$ except for $p = 11$ and $\mathrm{ord}_{11}(3^{10} - 1) = 2$. For $d = 1, -1, -3$, denote by X_d the sets of odd $\bar{\boldsymbol{Q}}_2$-valued Dirichlet characters associated to K_0 and $L_0 = L_{d,0}$ with $d = -1, -3$, respectively. For each odd prime number ℓ, let ω_ℓ be the Teichmüller character of conductor ℓ. Then we easily see that X_d are the sets of the characters ω_p^{2k+1}, $\omega_4\omega_p^{2k}$ and $\omega_3\omega_p^{2k}$ with $0 \leq k \leq (p-3)/2$, respectively for $d = 1, -1$ and -3. Let Y_d be the set of even characters $\xi = \chi^* = \omega_4\chi^{-1}$ with $\chi \in X_d$; namely Y_d is the set of duals of the characters in X_d. We denote by Z_d a complete set of representatives of the $\boldsymbol{Q}_2$-conjugacy classes of the characters in Y_d. Let ξ be an arbitrary even character of the 1st kind at the prime 2. (For instance, those characters in Y_d.) Letting $\xi^* = \omega_4\xi^{-1} = \chi$ be the dual character of ξ, we put $G_\xi(T) = H_{\chi^{-1}}(T) \in \mathcal{O}_\xi[[T]]$ with $\mathcal{O}_\xi = \mathcal{O}_\chi$. Here, $H_{\chi^{-1}}(T)$ is the power series associated by (2) to $L_2(s, \omega_4\chi^{-1})$. Let λ_ξ be the lambda invariant of the power series $G_\xi(T)$, and $n_\xi = [\boldsymbol{Q}_2(\xi) : \boldsymbol{Q}_2]$ where $\boldsymbol{Q}_2(\xi)$ is the field generated by the values of ξ. It is shown in Greenberg [4, Theorem 2] that

$$\lambda_0^- = \sum_{\xi \in Z_1} n_\xi \cdot \lambda_\xi \quad \text{and} \quad \lambda_0^* = \sum_{\xi \in Z_d} n_\xi \cdot \lambda_\xi \tag{10}$$

using the analytic class number formula. Here, ξ runs over the nontrivial characters $\xi \in Z_d$. Similarly, for the case $d = -3$ and $p = 11$, we have

$$\lambda_1^* = \lambda_0^* + \sum_{\xi \in Z_d} 10\, n_\xi \cdot \lambda_{\xi\psi_{11}}, \tag{11}$$

where ψ_{11} is an even character of conductor 11^2 and order 11. (Here, the factor 10 in the right-hand side of (11) is the degree $[\boldsymbol{Q}_2(\zeta_{11}) : \boldsymbol{Q}_2]$.)

We write

$$G_\xi(T) = \sum_{i \geq 0} c_{\xi,i} T^i \in \mathcal{O}_\xi[[T]].$$

By a theorem of Ferrero and Washington [14, Theorem 7.15], $G_\xi(T)$ is not divisible by a prime element of $\mathcal{O}_\xi$. Hence, the λ-invariant λ_ξ of G_ξ is the smallest integer i with $c_{\xi,i} \in \mathcal{O}_\xi^\times$. To compute this invariant, the following approximation formula is useful:

$$G_\xi(T) \equiv -\frac{1}{2^{j+3}c} \sum_{a=1}^{2^{j+2}c} a\xi^*(a)^{-1}(1+T)^{-\gamma_j(a)} \tag{12}$$

modulo $(1+T)^{2^j} - 1$ for $j \geq 0$. (See [14, §7].) Here, c is the odd part of the conductor of ξ, and a runs over the odd integers with $1 \leq a \leq 2^{j+2}c$ and $(a,\, c) = 1$, and $\gamma_j(a)$ is the integer satisfying $0 \leq \gamma_j(a) < 2^j$ and $(1+4c)^{\gamma_j(a)} \equiv a$ or $-a \bmod 2^{j+2}$ according as $a \equiv 1$ or -1 modulo 4.

For each $\xi \in Z_d$, we computed the right-hand side of (12) with $j = 0, 1, 2, 3, \cdots$ until we obtained λ_ξ. It turned out that the maximal value of λ_ξ in the range $p \leq 599$ is 64 ($< 2^7$), which is attained when $p = 257$ and $d = -3$, and our computation was completed with $j = 7$. We obtain the value λ_0^- and λ_0^* by (10). All the pairs (p, λ_0^-) (resp. (p, λ_0^*) for $d = -1, -3$) with $p \leq 599$ and $\lambda_0^- \geq 1$ (resp. $\lambda_0^* \geq 1$) are given in Table 3 (resp. Table 4, 5) at the end of this paper. We denote by d_ξ the order of $\xi \in Z_d$. In the tables, we also give the values d_ξ, n_ξ and λ_ξ for each $\xi \in Z_d$ with $\lambda_\xi \geq 1$. Further, we have

Proposition 3 *For those odd prime numbers p with $p \leq 599$ which is not contained in Table 3 (resp. Table 4, 5), we have $\lambda_0^- = 0$ (resp. $\lambda_0^* = 0$).*

Proposition 4 *For $p = 11$ and $d = -3$, we have $\lambda_1^* = 10$, $\lambda_0^* = 0$.*

Proof. We already know that $\lambda_0^* = 0$. For each $\xi \in Z_{-3}$, we computed the invariant $\lambda_{\xi\psi_{11}}$ using (12). Then, we have that $\lambda_{\xi\psi_{11}} = 1$ when $\xi = \omega_4\omega_3$

and $\lambda_{\xi\psi_{11}} = 0$ otherwise, and we obtain $\lambda_1^* = 10$ by (11). $\square$

Our computation was carried out with UBASIC [12] on a PC with 2.8 GHz Intel Core 2 Duo and 3 GB memory. The total time of computation was about one and a half hour.

Some relations on λ-invariants in the tables are explained by Kida's formulas. For example, we can compute the λ-invariant of an imaginary abelian 2-extension M of $\boldsymbol{Q}$ such that M contains an imaginary quadratic field F in the following way. First, by [10, Theorem 1], we have

$$\lambda^-(F) - \delta(F) = s_j(F/\boldsymbol{Q}) - 1 \tag{13}$$

for $j \gg 0$, where $\delta(F)$ is 1 or 0 according to whether or not F_∞ contains ζ_4, and $s_j(F/\boldsymbol{Q})$ is the number of finite primes of F_j which are ramified in $F_j/\boldsymbol{Q}_j$ and do not divide 2. Next, by [10, Theorem 3], we have

$$\begin{aligned} &\lambda^-(M) - \delta(M) \\ &\qquad = [M_\infty : F_\infty]\{\lambda^-(F) - \delta(F)\} + \textstyle\sum(e(\mathfrak{P}) - 1) - \sum(e(\mathfrak{P}_+) - 1), \end{aligned} \tag{14}$$

where $e(\mathfrak{P})$ (resp. $e(\mathfrak{P}_+)$) is the ramification index in M_∞/F_∞ (resp. M_∞^+/F_∞^+) of a finite prime $\mathfrak{P}$ of M_∞ (resp. $\mathfrak{P}_+$ of M_∞^+) and the sum are taken over all $\mathfrak{P}$ and $\mathfrak{P}_+$ which do not divide 2 respectively.

By (13), we have

$$\begin{aligned} \lambda^-(\boldsymbol{Q}(\sqrt{d})) &= 1 + 0 - 1 = 0 \quad \text{for } d = -1, \\ \lambda^-(\boldsymbol{Q}(\sqrt{d})) &= 0 + 1 - 1 = 0 \quad \text{for } d = -3. \end{aligned}$$

Applying (14) to $p = 257$ and $M/\boldsymbol{Q}(\sqrt{d})$ with $M \subseteq L_0$, we have

$$\begin{aligned} \lambda^-(M) &= 1 + 2^i \cdot (0-1) + 128 \cdot (2^i - 1) - 64 \cdot (2^i - 1) \\ &= 63 \cdot 2^i - 63 && \text{for } d = -1, \\ \lambda^-(M) &= 0 + 2^i \cdot (0-0) + 128 \cdot (2^i - 1) - 64 \cdot (2^i - 1) \\ &= 64 \cdot 2^i - 64 && \text{for } d = -3, \end{aligned}$$

where $2^i = [M : \boldsymbol{Q}(\sqrt{d})]$ and $0 \le i \le 7$. The above computation and (10) imply $\lambda_\xi = 63$ (resp. 64) for $p = 257$ and any $\xi \in Z_{-1}$ (resp. Z_{-3}), which agrees with the value in Table 4 (resp. Table 5).

Table 3: Iwasawa invariants $\lambda_0^- \geq 1$, ω_p^{2k+1}-part

p	d_ξ	n_ξ	λ_ξ	λ_0^-	p	d_ξ	n_ξ	λ_ξ	λ_0^-
7	2	1	1	1	283	6	2	1	2
17	16	8	3	24	307	6	2	1	2
23	2	1	1	1	311	2	1	1	
29	28	6	1	6		62	5	1	
31	2	1	7			62	5	1	11
	6	2	1	9	313	8	4	1	4
41	8	4	1	4	331	22	10	1	10
43	6	2	1	2	337	16	8	3	
47	2	1	3	3		336	48	1	72
71	2	1	1	1	349	12	4	1	4
73	8	4	1	4	353	32	16	7	112
79	2	1	3	3	359	2	1	1	1
89	8	4	1	4	367	2	1	3	3
97	32	16	7	112	373	124	10	1	10
103	2	1	1	1	383	2	1	31	31
109	12	4	1	4	397	12	4	1	
113	16	8	3			36	12	1	16
	112	24	1	48	401	16	8	3	24
127	2	1	31		409	8	4	1	4
	6	2	1		421	60	8	1	8
	18	6	1	39	431	2	1	3	
137	8	4	1	4		10	4	1	7
151	2	1	1		433	16	8	3	
	10	4	1	5		48	16	1	40
157	12	4	1	4	439	2	1	1	
163	6	2	2	4		6	2	1	3
167	2	1	1	1	449	64	32	15	480
191	2	1	15	15	457	8	4	1	
193	64	32	15	480		24	8	1	12
197	28	6	2	12	463	2	1	3	
199	2	1	1	1		14	3	3	12
223	2	1	7		479	2	1	7	7
	6	2	1	9	487	2	1	1	1
229	12	4	1	4	491	14	3	1	
233	8	4	1	4		14	3	1	6
239	2	1	3		499	6	2	1	2
	14	3	1	6	503	2	1	1	1
241	16	8	3		521	8	4	1	4
241	80	32	1	56	547	6	2	2	4
251	10	4	1	4	569	8	4	1	4
257	256	128	63	8064	571	10	4	1	4
263	2	1	1	1	577	64	32	15	480
271	2	1	3	3	593	16	8	3	24
277	12	4	1	4	599	2	1	1	1
281	8	4	1	4					

Table 4: Iwasawa invariants $\lambda_0^* \geq 1$ $(d=-1)$, $\omega_4\omega_p^{2k}$-part

p	d_ξ	n_ξ	λ_ξ	λ_0^*	p	d_ξ	n_ξ	λ_ξ	λ_0^*	p	d_ξ	n_ξ	λ_ξ	λ_0^*
17	2	1	3		239	7	3	1	3	401	2	1	3	
	4	2	3		241	2	1	3			4	2	3	
	8	4	3	21		4	2	3			8	4	3	21
29	7	3	1			5	4	1		409	2	1	1	
	14	3	1	6		8	4	3			4	2	1	3
31	3	2	1	2		10	4	1		421	15	4	1	
41	2	1	1			20	8	1			30	4	1	8
	4	2	1	3		40	16	1	53	431	5	4	1	4
43	3	2	1	2	251	5	4	1	4	433	2	1	3	
73	2	1	1		257	2	1	63			3	2	1	
	4	2	1	3		4	2	63			4	2	3	
89	2	1	1			8	4	63			6	2	1	
89	4	2	1	3		16	8	63			8	4	3	
97	2	1	7			32	16	63			12	4	1	
	4	2	7			64	32	63			24	8	1	37
	8	4	7			128	64	63	8001	439	3	2	1	2
	16	8	7	105	277	3	2	1		449	2	1	15	
109	3	2	1			6	2	1	4		4	2	15	
	6	2	1	4	281	2	1	1			8	4	15	
113	2	1	3			4	2	1	3		16	8	15	
	4	2	3		283	3	2	1	2		32	16	15	465
	7	3	1		307	3	2	1	2	457	2	1	1	
	8	4	3		311	31	5	1			3	2	1	
	14	3	1			31	5	1	10		4	2	1	
	28	6	1		313	2	1	1			6	2	1	
	56	12	1	45		4	2	1	3		12	4	1	11
127	3	2	1		331	11	10	1	10	463	7	3	3	9
	9	6	1	8	337	2	1	3		491	7	3	1	
137	2	1	1			4	2	3			7	3	1	6
	4	2	1	3		8	4	3		499	3	2	1	2
151	5	4	1	4		21	6	1		521	2	1	1	
157	3	2	1			42	6	1			4	2	1	3
	6	2	1	4		84	12	1		547	3	2	2	4
163	3	2	2	4		168	24	1	69	569	2	1	1	
193	2	1	15		349	3	2	1			4	2	1	3
	4	2	15			6	2	1	4	571	5	4	1	4
	8	4	15		353	2	1	7		577	2	1	15	
	16	8	15			4	2	7			4	2	15	
	32	16	15	465		8	4	7			8	4	15	
197	7	3	2		353	16	8	7	105		16	8	15	
	14	3	2	12	373	31	5	1			32	16	15	465
223	3	2	1	2		62	5	1	10	593	2	1	3	
229	3	2	1		397	3	2	1			4	2	3	
	6	2	1	4		6	2	1			8	4	3	21
233	2	1	1			9	6	1						
	4	2	1	3		18	6	1	16					

Table 5: Iwasawa invariants $\lambda_0^* \geq 1$ ($d = -3$), $\omega_3\omega_p^{2k}$-part

p	d_ξ	n_ξ	λ_ξ	λ_0^*	p	d_ξ	n_ξ	λ_ξ	λ_0^*	p	d_ξ	n_ξ	λ_ξ	λ_0^*
5	2	1	1	1	197	2	1	1		401	2	1	4	
13	2	1	1	1		14	3	2			4	2	4	
17	2	1	4			14	3	2	13		8	4	4	28
	4	2	4		223	6	2	1	2	409	2	1	2	
	8	4	4	28	229	2	1	1			4	2	2	6
29	2	1	1			6	2	1		421	2	1	1	
	14	3	1			6	2	1	5		30	4	1	
	14	3	1	7	233	2	1	2			30	4	1	9
31	6	2	1	2		4	2	2	6	431	10	4	2	8
37	2	1	1	1	239	14	3	1	3	433	2	1	4	
41	2	1	2		241	2	1	4			4	2	4	
	4	2	2			4	2	4			6	2	1	
	10	4	1			8	4	4			6	2	1	
	10	4	1			10	4	1			8	4	4	
	20	8	1	22		10	4	1			12	4	1	
43	6	2	1	2		20	8	1			24	8	1	44
53	2	1	1	1		40	16	1	60	439	6	2	2	4
61	2	1	1		251	10	4	1	4	449	2	1	16	
	6	2	1		257	2	1	64			4	2	16	
	6	2	1	5		4	2	64			8	4	16	
67	6	2	1	2		8	4	64			16	8	16	
73	2	1	2			16	8	64			32	16	16	496
	4	2	2			32	16	64		457	2	1	2	
	6	2	1			64	32	64			4	2	2	
	6	2	1			128	64	64	8128		6	2	1	
	12	4	1	14	269	2	1	1	1		6	2	1	
89	2	1	2		271	6	2	1			12	4	1	14
	4	2	2	6		18	6	1	8	461	2	1	1	1
97	2	1	8		277	2	1	1		463	14	3	3	9
	4	2	8			6	2	1		491	10	4	1	
	8	4	8			6	2	1	5		14	3	1	
	16	8	8	120	281	2	1	2			14	3	1	10
101	2	1	1	1		4	2	2	6	499	6	2	2	4
103	6	2	1	2	283	6	2	1	2	509	2	1	1	1
109	2	1	1		293	2	1	1	1	521	2	1	2	
	6	2	1		307	6	2	2			4	2	2	6
	6	2	1	5		18	6	1	10	523	6	2	1	
113	2	1	4		311	62	5	1			18	6	1	8
	4	2	4			62	5	1	10	541	2	1	1	1
	8	4	4		313	2	1	2		547	6	2	3	
	14	3	1			4	2	2	6		26	12	1	
	14	3	1		317	2	1	1	1		78	12	1	
	28	6	1		331	22	10	1	10		78	12	1	42
	56	12	1	52	337	2	1	4		557	2	1	1	1
127	6	2	1			4	2	4		569	2	1	2	
	18	6	1	8		8	4	4			4	2	2	6
137	2	1	2			42	6	1		571	10	4	1	4
	4	2	2	6		42	6	1		577	2	1	16	
149	2	1	1	1		84	12	1			4	2	16	
151	6	2	1			168	24	1	76		6	2	1	
	10	4	1	6	349	2	1	1			6	2	1	
157	2	1	1			6	2	1			8	4	16	
	6	2	1			6	2	1	5		12	4	1	
	6	2	1	5	353	2	1	8			16	8	16	
163	6	2	2	4		4	2	8			24	8	1	
173	2	1	1	1		8	4	8			32	16	16	
181	2	1	1	1		16	8	8	120		48	16	1	
193	2	1	16		367	6	2	1	2		96	32	1	560
	4	2	16		373	2	1	1		593	2	1	4	
	6	2	1			62	5	1			4	2	4	
	6	2	1			62	5	1	11		8	4	4	28
	8	4	16		389	2	1	1	1					
	12	4	1		397	2	1	1						
	16	8	16			6	2	1						
	24	8	1			6	2	1						
	32	16	16			18	6	1						
	48	16	1			18	6	1	17					
	96	32	1	560										

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