A generalized problem associated to the Kummer-Vandiver conjecture

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Abstract

In order to discuss the validity of the Kummer-Vandiver conjecture, we consider a generalized problem associated to the conjecture. Let p be an odd prime number and ζ_p a primitive p-th root of unity. Using new programs, we compute the Iwasawa invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$ in the range $|d| < 200$ and $200 < p < 1,000,000$. From our data, the actual numbers of exceptional cases seem to be near the expected numbers for $p < 1,000,000$. Moreover, we find a few rare exceptional cases for $|d| < 10$ and $p > 1,000,000$. We give two partial reasons why it is difficult to find exceptional cases for $d = 1$ including counter-examples to the Kummer-Vandiver conjecture.

Key words: Iwasawa invariants, Kummer-Vandiver conjecture, ideal class group
2020 Mathematics Subject Classification: Primary 11R23; Secondary 11R18, 11R29, 11R70

1 Introduction

Let p be an odd prime number and K a finite extension of $\mathbf{Q}$. K_∞ denotes the cyclotomic $\mathbf{Z}_p$-extension of K. Let K_n be its n-th layer and $A_n = A_n(K)$ the p-part of the ideal class group of K_n.

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[*] ORCID:0000-0003-3928-9481 The author was partially supported by JSPS KAKENHI Grant Number JP17K05176 and JP20H00115.

First, let K be the p-cyclotomic field $\mathbf{Q}(\zeta_p)$, then $K_n = \mathbf{Q}(\zeta_{p^{n+1}})$. Let $\omega = \omega_p$ be the Teichmüller character $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}_p$ such that $\omega(a) \equiv a \bmod p$. We identify $\Delta = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$ with $(\mathbf{Z}/p\mathbf{Z})^\times$. Put $e_{\omega^k} = \frac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\omega^k(\delta)\delta^{-1}$ the idempotent of the group ring $\mathbf{Q}_p[\Delta]$. Then we have

$$A_n = \bigoplus_{k:even} e_{\omega^k}A_n \oplus \bigoplus_{p-k:odd} e_{\omega^{p-k}}A_n,$$

where k is an even integer with $2 \leq k \leq p-1$. Let A_n^+ (resp. A_n^-) be the even part (resp. odd part). Let r_p be the irregularity index, i.e., the number of irregular pairs (p,k). Irregular pairs have been computed by Kummer, Vandiver, D.H. Lehmer, E. Lehmer, Selfridge, Nicol, Pollack, Johnson, Wada, Wagstaff, Tanner, Ernvall, Metsänkylä, Buhler, Crandall, Sompolski, Shokrollahi, Hart, Harvey and Ong. These computations had been connected with verification of Fermat's last theorem. However, even after the proof was completed by Wiles, they are still interesting because they give us concrete knowledge of the ideal class group of cyclotomic fields. In [1, 2, 5] etc., for any prime number $p < 2^{31} = 2,147,483,648$, it has been verified that

$$A_n^+ = \{0\} \text{ and } A_n^- \simeq (\mathbf{Z}/p^{n+1}\mathbf{Z})^{r_p} \text{ for all } n \geq 0.$$

The former statement is called the Kummer-Vandiver conjecture. We have a naive explanation of the fact that we have not been able to find any counter-example. If we follow the heuristic argument of [15, pp.158–159], we can expect that the number of exceptions to the Kummer-Vandiver conjecture for $x_0 \leq p \leq x_1$ is approximately $(\log\log x_1 - \log\log x_0)/2$. Then, $(\log\log 2^{31} - \log\log 37)/2 = 0.891756\cdots$ is probably too small to find one counter-example, where 37 is the smallest irregular prime number. Furthermore, the expected number would not be exact, because there are some effects on ideal class groups from an upper bound for the numerator of the Bernoulli number or the K-groups (cf. [10]). If there are another strong effects, the actual number could be much less than the above number. In order to study the heuristic, we consider the following generalized problem.

Problem 1.1 *Let F be an abelian extension of $\mathbf{Q}$. Let $N_F(x)$ be the number of prime numbers p such that $A_0(F(\zeta_p)^+) \neq \{0\}$ for $p \leq x$, where $F(\zeta_p)^+$ is the maximal real subfield of $F(\zeta_p)$. Is $N_F(x)$ bounded as $x \to \infty$? If it is not so, give an approximate function for $N_F(x)$.*

The Kummer-Vandiver conjecture claims that $N_{\mathbf{Q}}(x) = 0$ for all x, which is much stronger than its boundedness.

In this paper, following [11, 12, 13, 14], we study the above problem when F is $\mathbf{Q}$ or a quadratic field, because they are easy to be compared. Let χ be the Dirichlet character associated to F and f_χ its conductor. The main purpose of the paper is to find exceptional cases associated to the $\chi\omega^k$-part in order to argue about the expected number. We actually computed the Iwasawa invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$ in the range $|d| < 200$ and $200 < p < 1,000,000$ by using new programs, where $d = d_\chi = \chi(-1)f_\chi$. From our data, the actual number seems to be near the expected number in the range. Moreover, we found a few rare exceptional cases for $|d| < 10$ and $1,000,000 < p < 20,000,000$.

Our computations are executed in $O((f_\chi p)^{1+\epsilon})$ bit operations. See [12, 14] on the relation between these Iwasawa invariants and the higher K-groups of the integer ring of $\mathbf{Q}(\sqrt{d})$.

2 Iwasawa invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$

Let χ be the trivial character or a quadratic Dirichlet character conductor $f = f_\chi$ and p an odd prime number such that p does not divide f. Put $d = d_\chi = \chi(-1)f_\chi$, $K = \mathbf{Q}(\sqrt{d_\chi}, \zeta_p)$, then $K_n = \mathbf{Q}(\sqrt{d_\chi}, \zeta_{p^{n+1}})$. Let A_n be the p-part of the ideal class group of K_n.

Put $\Gamma = \mathrm{Gal}(K_\infty/K)$, $\Delta = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$ and $e_\psi = \frac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\psi(\delta)\delta^{-1}$ for a character ψ of Δ. We put $f_0 = fp$ and identify Δ with a subquotient of $(\mathbf{Z}/f_0\mathbf{Z})^\times$ in the ordinary way. For a $\mathbf{Z}_p[\Delta]$-module A, A^ψ denotes $e_\psi A$. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ be the Iwasawa invariants associated to A_n^ψ, i.e.,

$$\sharp A_n^\psi = p^{\lambda_p(\psi)n + \mu_p(\psi)p^n + \nu_p(\psi)}$$

for sufficiently large n. By Ferrero-Washington's theorem, we have $\mu_p(\psi) = 0$ for all p and ψ.

Assume that ψ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the p-adic L-function is defined as follows. Let $L_p(s, \psi)$ be the p-adic L-function constructed by [8]. By [7, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi((1+f_0)^{1-s}-1) = L_p(s,\psi)$ for all $s \in \mathbf{Z}_p$ if $\psi \neq \chi^0$. By [3], p does not divide $G_\psi(T)$. By the p-adic Weierstrass preparation theorem, we can uniquely write $G_\psi(T) = g_\psi(T)u_\psi(T)$, where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Similarly we can define $g_\psi^*(T) \in \mathbf{Z}_p[T]$ from $G_\psi^*(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi^*((1+f_0)^s - 1) = L_p(s, \psi)$. Put

$$\tilde{\lambda}_p(\psi) = \deg g_\psi(T) = \deg g_\psi^*(T).$$

Put $f_n = f_0 p^n$ and let $\gamma \in \Gamma \simeq \mathrm{Gal}(\cup_{n\geq 0}\mathbf{Q}(\zeta_{f_n})/\mathbf{Q}(\zeta_{f_0}))$ be the generator of Γ such that $\zeta_{f_n}^{\tilde{\gamma}} = \zeta_{f_n}^{1+f_0}$ for all $n \geq 0$. As usual, we can identify the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the formal power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma = 1 + T$. By this identification, we can consider a $\mathbf{Z}_p[[\Gamma]]$-module as a Λ-module. For a finitely generated torsion Λ-module A, we define the Iwasawa polynomial $\mathrm{char}_\Lambda(A)$ to be the characteristic polynomial of the action T on $A \otimes \mathbf{Q}_p$ (cf. [15, §13]). Let L_n be the maximal unramified abelian extension of K_n and M_n the maximal abelian extension of K_n unramified outside p. By the class field theory, we have $A_n \simeq \mathrm{Gal}(L_n/K_n)$. Set $L_\infty = \cup_{n\geq 0}L_n$, $M_\infty =$

$\cup_{n\geq 0}M_n$, $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$ and $Y_\infty = \mathrm{Gal}(M_\infty/K_\infty)$. By the Iwasawa main conjecture proved by [4, 9], $\mathrm{char}_\Lambda(X_\infty^{\psi^{-1}\omega}) = g_\psi^*(T)$ and $\mathrm{char}_\Lambda(Y_\infty^\psi) = g_\psi(T)$.

In the following, we assume that

$$\psi = \chi\omega^k \text{ is even, and } \psi^* = \psi^{-1}\omega = \chi\omega^{p-k} \text{ is odd}$$

with $2 \leq k \leq p-2$. Since p does not divide f,

$$\text{(C)} \qquad \psi(p) \neq 1 \text{ and } \psi^*(p) \neq 1.$$

By (C), we have that $A_n^\psi \simeq X_\infty^\psi/\omega_n X_\infty^\psi$ and $A_n^{\psi^*} \simeq X_\infty^{\psi^*}/\omega_n X_\infty^{\psi^*}$, where $\omega_n = (1+T)^{p^n}-1$ (cf. [6, Lemma 3 and Remark 4]). Moreover, if A_0^ψ is trivial, we have $\lambda(\psi) = \nu(\psi) = 0$, $X_\infty^\psi = \{0\}$, $Y_\infty^\psi \simeq \Lambda/(g_\psi(T))$ and $X_\infty^{\psi^*} \simeq \Lambda/(g_\psi^*(T))$. Put $a_0 = a_0(\psi) = L_p(1,\psi) = G_\psi(0)$ and $b_0 = b_0(\psi) = L_p(0,\psi) = G_\psi^*(0)$.

We call $(p, \chi\omega^k)$ exceptional pairs when one of the following conditions holds: $[\nu] : \nu(\chi\omega^k) > 0$, $[a_0] : v_p(a_0) > 1$, $[b_0] : v_p(b_0) > 1$ or [lmd] : $\tilde{\lambda}(\chi\omega^k) > 1$. In [11, 12, 13, 14], we computed exceptional pairs for $|d| < 200$ and $p < 200{,}000$. By further computation, we obtain the following.

Proposition 2.1 *For $|d| < 200$ and $200{,}000 < p < 1{,}000{,}000$, all exceptional pairs $(p, \chi\omega^k)$ are given in Table 1.*

Table 1: Exceptional pairs for $|d| < 200$ and $200{,}000 < p < 1{,}000{,}000$.

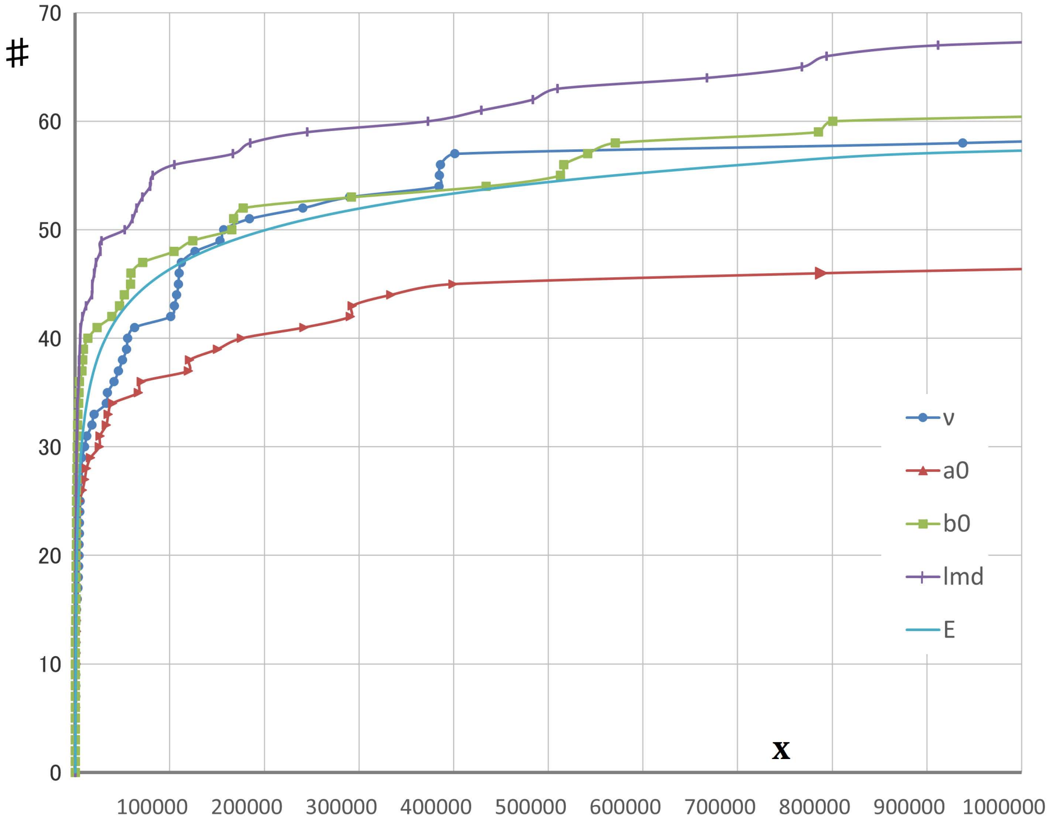
$[\nu]$			$[a_0]$		
p	k	d	p	k	d
240571	146919	-43	241817	134764	53
289897	186889	-131	290627	50599	-151
384487	13724	161	292801	242013	-104
384847	226771	-143	333581	180787	-71
386119	263582	149	399181	1683	-4
401321	205162	185	788687	186548	141
937943	11057	-167			
$[\nu]$			$[a_0]$		
p	k	d	p	k	d
292157	48631	-111	245177	59489	-20
434389	402352	93	312089	21817	-159
512891	91273	-120	372871	329947	-104
516323	63368	136	429427	61972	92
541759	285435	-71	483773	271222	33
570781	405689	-52	509581	402749	-195
785303	359267	-67	667727	487990	113
800447	136068	161	768013	754145	-111
			794141	494244	165
			911831	821980	165

In order to study Problem 1.1 efficiently, we consider the following problem.

Problem 2.1 *Let X be a set of primitive Dirichlet characters. Let $N^{[\nu]}_{X,x_0}(x)$ be the number of pairs $(p, \chi\omega^k)$ such that $\nu(\chi\omega^k) > 0$ for $\chi \in X$ in the range $x_0 \leq p \leq x$. We similarly define $N^{[a_0]}_{X,x_0}(x)$, $N^{[b_0]}_{X,x_0}(x)$ and $N^{[lmd]}_{X,x_0}(x)$. Are they bounded as $x \to \infty$? If they are not so, give approximate functions for them.*

In Figure 1, we compare actual numbers of exceptional pairs in the range $f_\chi < 200$ and $200 < p \leq x$ with the expected number $E(x) = 123 \sum_{200<p\leq x} \frac{p-3}{2} \left(\frac{1}{p}\right)^2$, where 123 is the number of the trivial or quadratic characters χ with $f_\chi < 200$. From our data, actual numbers still seem to be near the expected number.

Figure 1: Actual numbers and the expected number $(200 < p < 1,000,000)$.



For $|d| < 10$, we obtain the following by further computation.

Proposition 2.2 *For $|d| < 10$ and $10 < p < 20,000,000$, all exceptional pairs $(p, \chi\omega^k)$ are given in Table 2.*

Table 2: Exceptional pairs for $|d| < 10$ and $10 < p < 20,000,000$.

[ν]			[a_0]		
p	k	d	p	k	d
379	317	-4	59	36	8
34301	114	8	1381	609	-4
157229	140434	8	399181	1683	-4
			5911877	1629992	5
[b_0]			[lmd]		
p	k	d	p	k	d
173	97	-7	23	11	-8
257	101	-3	1151	842	8
2221	1600	8	3613	1147	-7
4953979	1174520	5	27791	11840	8
			1744817	928867	-3

Remark 2.1 In [14, Proposition 2], we reported that there is only one exceptional pair $(399181, \chi_{-4}\omega^{1683})$ in the range $|d| < 10$ and $200,000 < p < 1,000,000$, which is included in [lmd] by mistake.

3 A conjecture on the number of exceptional pairs

We give two partial reasons why it is difficult to find exceptional pairs for $\chi = \chi^0$, i.e., $d = 1$. The first partial reason is the fact that the expected number for χ^0 is smaller than those for the other characters. Let $r_{p,\chi}$ be the number of pairs $(p, \chi\omega^k)$ such that $\tilde{\lambda}_p(\chi\omega^k) > 0$. Then we have $0 \leq r_{p,\chi} \leq (p-3)/2$. The distribution of the number of p such that $r_{p,\chi} = r$ for each χ in the range $200 < p < 20,000,000$ is similar to that for χ^0. However, the distribution for χ in the range $10 < p < 200$ is not always similar to that for χ^0, which affects expected numbers. For each χ, put $E_\chi(x) = \displaystyle\sum_{10<p\leq x,\ p:\text{prime}} \frac{r_{p,\chi}}{p}$. The following table and figures show differences among $E_\chi(x)$s.

Table 3: Expected numbers of exceptional pairs up to $x = 10^n$.

$x \backslash d$	1	5	8	-3	-4	-7	-8
10^2	0.06880	0.19020	0.28139	0.12712	0.19738	0.17111	0.15889
10^3	0.24468	0.37505	0.42811	0.25481	0.33259	0.38703	0.33137
10^4	0.38967	0.51498	0.56373	0.39079	0.47707	0.52158	0.47146
10^5	0.49996	0.62089	0.67539	0.50485	0.58772	0.63171	0.58758
10^6	0.59054	0.71359	0.76666	0.59531	0.67887	0.72346	0.67885
10^7	0.66785	0.79089	0.84353	0.67206	0.75614	0.80081	0.75607

Figure 2: Expected numbers of exceptional pairs up to $10,000$.

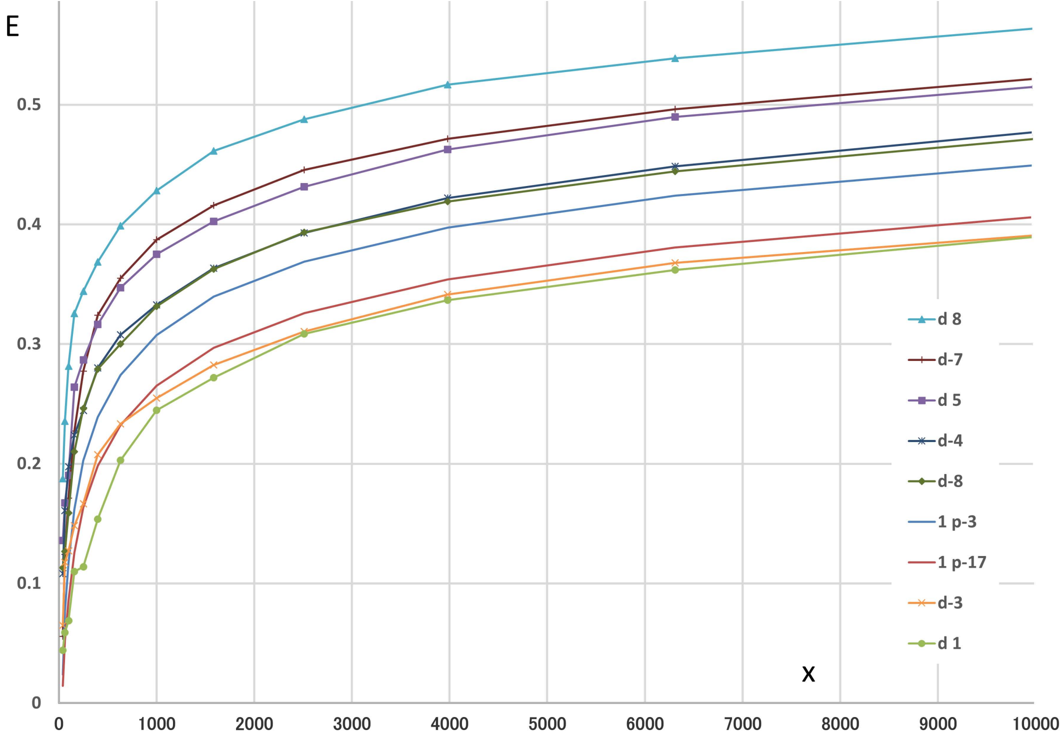
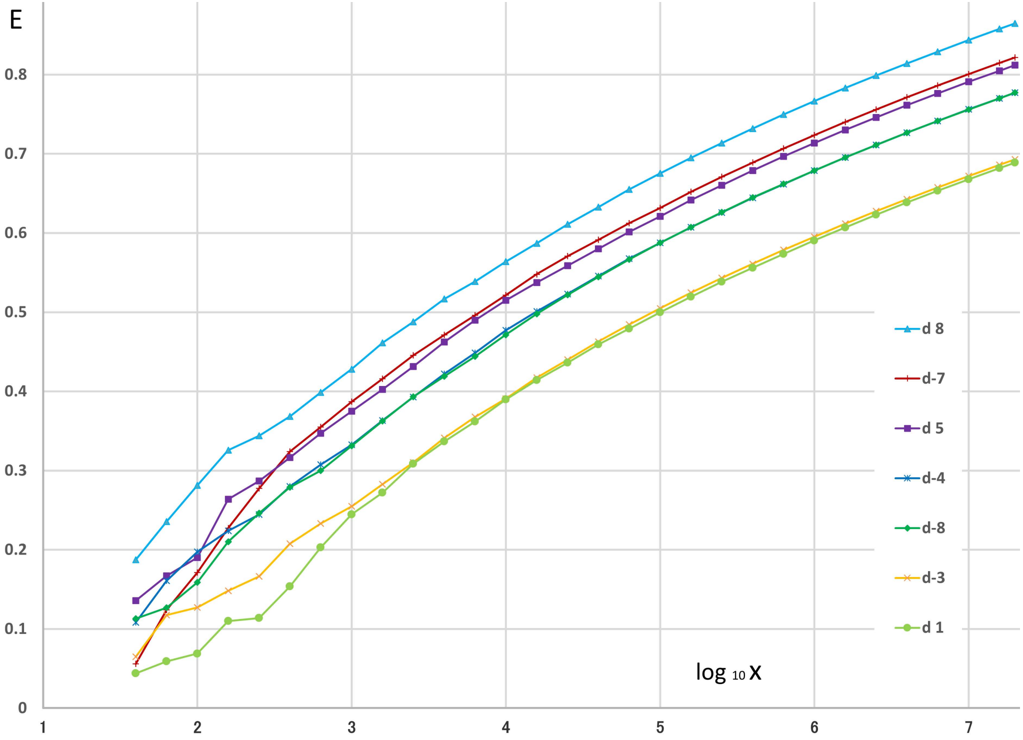


Figure 3: Expected numbers of exceptional pairs up to $20,000,000$.



In Figures 2 and 3, we compare $E_\chi(x)$s with $E'(x) = \displaystyle\sum_{37 \le p \le x} \frac{p-3}{2p^2}$ and $E''(x) = \displaystyle\sum_{37 \le p \le x} \frac{p-17}{2p^2}$, where $E''(x)$ is defined by considering trivialities of $K_4(\mathbf{Z})$ and numerators of Bernoulli numbers B_{2i} for i =1–5 and 7 (see [11, §4.2]).

Table 4 shows the total number N_d of exceptional pairs in $200 < p < 1{,}000{,}000$ for each d, the distribution (%) and a Poisson distribution with $\lambda = 4 \displaystyle\sum_{200 < p < 1{,}000{,}000} \frac{p-3}{2p^2} = 1.8721\cdots$. Since there are 23 d's with $N_d = 0$ among 123 d's, it is not very unusual that $N_1 = 0$. This is the second partial reason.

Table 4: The total number of exceptional pairs for d in $200 < p < 1{,}000{,}000$.

N_d	$\sharp$	%	Poisson	d
0	23	18.699	15.380	1,5,-11,13,-15,24,-24,-35,-40,-56,-68,69,-87,-95, -103,-107,-123,129,-132,-148,-164,173,-191
1	35	28.455	28.793	-3,-7,12,17,-23,29,-31,-55,-59,65,73,77,-83,85,88, 89,92,93,101,104,-119,120,137,-143,145,-155,168, 172,-179,-187,188,-195,197,-199
2	30	24.390	26.952	-8,-20,21,33,37,40,41,-43,44,-47,57,60,61,76,-84, -88,109,-184,-111,113,-115,136,140,152,-159,-163, 177,-183,184,185,193
3	16	13.008	16.819	-4,28,-51,-52,56,97,105,-127,133,-136,-151,-152, 157,-167,-168,181
4	11	8.9431	7.8716	-39,53,-67,-91,-116,-120,-139,141,156,161,165
5	4	3.2520	2.9472	8,-71,-104,124
6	3	2.4390	0.9196	-19,-79,149
7	1	0.8130	0.2459	-131

From our data and computational results on p-adic L-functions, it is natural to consider the following conjecture.

Conjecture 3.1 *Let X be a set of primitive Dirichlet characters. For $\chi \in X$, let n_χ be the order of χ, $\mathcal{O}_{p,\chi}$ the integer ring of $\mathbf{Q}_p(\zeta_{n_\chi})$, and $\mathfrak{m}_{p,\chi}$ the maximal ideal of $\mathcal{O}_{p,\chi}$. For a sufficiently large number x_0, an approximate function of $N^{[\nu]}_{X,x_0}(x)$, $N^{[a_0]}_{X,x_0}(x)$, $N^{[b_0]}_{X,x_0}(x)$ and $N^{[lmd]}_{X,x_0}(x)$ is given by the sum of*

$$\begin{aligned}\sum_{x_0 \le p \le x} \frac{p-3}{2}\left(\frac{1}{\sharp(\mathcal{O}_{p,\chi}/\mathfrak{m}_{p,\chi})}\right)^2 &= \sum_{\substack{x_0 \le p \le x \\ p \equiv 1 \bmod n_\chi}} \frac{p-3}{2}\left(\frac{1}{p}\right)^2 + \sum_{\substack{x_0 \le p \le x \\ p \not\equiv 1 \bmod n_\chi}} \frac{p-3}{2}\left(\frac{1}{p^{f_{p,\chi}}}\right)^2 \\ &\approx \frac{1}{\varphi(n_\chi)} \sum_{x_0 \le p \le x} \frac{p-3}{2p^2} + O(1) \\ &\approx (\log\log x - \log\log x_0)/(2\varphi(n_\chi)),\end{aligned}$$

over $\chi \in X$, *where* $p^{f_{p,\chi}} = \sharp(\mathcal{O}_{p,\chi}/\mathfrak{m}_{p,\chi}) \geq p^2$.

Remark 3.1 By replacing $\mathbf{Z}_p$ by $\mathcal{O}_{p,\chi}$, we can define $N^{[\nu]}_{X,x_0}(x), \ldots$ and $N^{[lmd]}_{X,x_0}(x)$. If χ is not the trivial character nor a quadratic character, then $\varphi(n_\chi) \geq 2$ and the expected number is clearly smaller than $\displaystyle\sum_{x_0 \leq p \leq x} \frac{p-3}{2p^2}$. This is a reason why we first study characters with $n_\chi \leq 2$.

4 Computations of arithmetic elements

In order to study Iwasawa invariants, we compute the following arithmetic elements (see [12, §5]):
(I) the generalized Bernoulli numbers modulo p, i.e., $\sum_{k=0}^{p-3} B_{k,\chi} t^k/k! \bmod p$,
$(\mathrm{II})_n$ the Iwasawa polynomial $g_{\chi\omega^k}(T) \mod p^{n+1}$,
$(\mathrm{III})_n$ the special cyclotomic unit $(c_n^{\chi\omega^k})^{Y_n(T)}$ modulo a prime ideal $\mathfrak{L}_n$,
$(\mathrm{IV})_n$ the Gauss sum $g_0(N_{K_n/K_0}\mathfrak{L}_n)^{\chi\omega^{p-k}}$ modulo a prime ideal $\mathfrak{L}_0^*$, where $\mathfrak{L}_n$ (resp. $\mathfrak{L}_n^*$) is a prime ideal above $l = 1 + kf_n$ (resp. $l^* = 1 + k^*(2f_n l)$) of K_n.

From 2002 to 2007, we used 32-bit programs (bcn.c for even χ's and bcm.c for odd ones) for computations of (I), $(\mathrm{II})_1$ and $(\mathrm{III})_0$ in [11, 12, 13, 14]. These programs work when f_0 and $8(p-3)\log_{16}(2p^3)$ are smaller than 2^{31}. Therefore, for $f = 1$ (resp. 199), they do not work when $p > 15,000,000$ (resp. 11,000,000). Since 2015, inspired by [2], we have been used 64-bit programs (bcn64.c and bcm64.c), which work when p is smaller than 162 million. Further, the program is available to check Greenberg's conjecture by computing $(\mathrm{III})_1$. However, since it needs $O(f_1^{1+\epsilon})$ bit operations, we did not check it for $p > 100,000$ except for $(p,k,d) = (157229, 140434, 8)$.

Computation of $(\mathrm{IV})_0$ rigorously proves that the cyclotomic unit $c_0^{\chi\omega^k}$ is a p-th power element in K_0, i.e., $\nu_p(\chi\omega^k) > 0$. From 2002 to 2007, we used 32-bit programs (gauss.c for small f_0's and gaussd.c for large ones) for computation of $(\mathrm{IV})_0$, which work when $2l$ is smaller that 2^{31}. Therefore, for $f = 1$ (resp. 199), they do not work when $p > 110,000,000/k$ (resp. $550,000/k$) with k =2–24. In order to reduce memory usage, we use HDD and several auxiliary primes $l_i \approx 1000$ in gaussd.c, which slows down the computation. Since 2020, we have been used 64-bit programs (gauss64.c and gaussd64.c), which work when $2l$ and $8f_0 \log_{16}(2l^{*2}f_0)$ are smaller than 2^{63}. In order to reduce memory usage, we use HDD and several auxiliary primes $l_i \approx 100000$ in gaussd64.c.

Computations of (I), $(\mathrm{II})_1$, $(\mathrm{III})_0$ and $(\mathrm{IV})_0$ are executed in $O(f_0^{1+\epsilon})$ bit operations. In Table 5, we give the ratios of execution times for (I)–(IV) by a single thread program on a Linux PC, where $*$ means that we did not compute it for lack of RAM.

Table 5: The ratios of execution times.

p	157229	937943	999983	999983	19999873	19999999
d	8	-167	1	-199	1	-7
(I)	1	41	5.4	47	280	320
$(\mathrm{II})_1$	0.5	190	1.3	240	48	300
$(\mathrm{III})_0$	0.4	74	1.1	94	38	83
$(\mathrm{IV})_0$	10	$*$	-	-	-	-
$(\mathrm{IV})_{0d}$	44	9000	-	-	-	-

For computations of (I) and $(\mathrm{IV})_0$, we need large RAM for the FFT algorithm. Various methods in [5] will be useful for speed-up and reduction of memory usage in computations of (I), $(\mathrm{II})_1$ and $(\mathrm{III})_0$. It will be able to speed up computations of $(\mathrm{II})_1$ and $(\mathrm{III})_0$ by parallel computing with GPU.

The above programs and further data have been available in our web page: https://math0.pm.tokushima-u.ac.jp/˜hiroki/major/galois1-e.html. These data were obtained by six personal computers for several years. Though we computed twice for $|d| < 200$ and $p < 1,000,000$, we computed only once for $|d| < 10$ and $p < 20,000,000$. Hence, we deduce that there must be some errors and that some irregular pairs are missing in the latter computation. However, considering their ratio, we deduce that these possible errors do not affect Proposition 2.2.

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