Nontrivial zeros of a pairing of p-units in the $4p$-cyclotomic field

Hiroki SUMIDA-TAKAHASHI*, Naoki FURUYA and Kodai KITANO

Department of Mathematical Sciences,
Tokushima University,
Minamijosanjima-cho 2-1, Tokushima 770-8506, JAPAN
e-mail address : hirokit@tokushima-u.ac.jp

Abstract

We study a pairing of p-units in the $4p$-cyclotomic field, following results on the p-cyclotomic field by McCallum–Sharifi. We discuss the distribution of the number of nontrivial zeros for each prime number $p < 2^{16} = 65,536$ under a conjecture, which give a sufficient condition for Greenberg's generalized conjecture. We also explain rare zeros which do not appear in the p-cyclotomic field.

Key words: paring of p-units, K_2-group, Greenberg's generalized conjecture
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1 Introduction

We first explain the main purpose of the computation, i.e., Greenberg's generalized conjecture. Let k be a finite extension of the rational number field $\mathbb{Q}$ and p an odd prime number. Let $\tilde{k}$ be the maximal multiple $\mathbb{Z}_p$-extension of k with $\mathrm{Gal}(\tilde{k}/k) \simeq \mathbb{Z}_p^d$. Leopoldt's conjecture for k and p implies that $d = r_2(k) + 1$, where $r_2(k)$ is the number of complex places of k. Let k_n be the intermediate field in $\tilde{k}/k$ such that $\mathrm{Gal}(k_n/k) \simeq (\mathbb{Z}/p^n\mathbb{Z})^d$, A_n the p-part of the ideal class group of k_n, and $X_\infty = \varprojlim A_n$, where the inverse limit is taken with respect to norm maps. Further let γ_i $(1 \le i \le d)$ be the topological generator of $\mathrm{Gal}(\tilde{k}/k)$ with $\overline{\langle \gamma_1, \gamma_2, ..., \gamma_d \rangle} \simeq \mathbb{Z}_p^d$. We can consider X_∞ as a $\tilde{\Lambda} = \mathbb{Z}_p[[T_1, T_2, ..., T_d]]$-module by the action of $T_i = \gamma_i - 1$.

Conjecture 1.1 (Greenberg's generalized conjecture)**.** *For any k and p, X_∞ is a pseudo-null $\tilde{\Lambda}$-module, i.e.,* $\mathrm{ht}_{\tilde{\Lambda}}(\mathrm{Ann}_{\tilde{\Lambda}} X_\infty) \ge 2$.

Remark 1.1. When $d = 1$, k is a totally real number field, and k_∞ is the cyclotomic $\mathbb{Z}_p$-extension. By Iwasawa's class number formula, we have $\sharp A_n = p^{\lambda n + \mu p^n + \nu}$ for any sufficiently large n, where $\lambda = \lambda_p(k_\infty/k)$, $\mu_p(k_\infty/k) \in \mathbb{Z}_{\ge 0}$ and $\nu = \nu_p(k_\infty/k) \in \mathbb{Z}$ are the Iwasawa invariants. The above conjecture implies that X_∞ is finite, i.e., $\lambda_p(k_\infty/k) = \mu_p(k_\infty/k) = 0$, which is called Greenberg's conjecture for the Iwasawa invariants of totally real number fields.

Remark 1.2. For any $\mathbb{Z}_p^r$-extension, it is shown that X_∞ is a finitely generated torsion $\Lambda = \mathbb{Z}_p[[T_1, T_2, ..., T_r]]$-module in [2] under some assumptions, which is known to be unnecessary.

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Hence we have $\mathrm{ht}_\Lambda(\mathrm{Ann}_\Lambda X_\infty) \geq 1$ in general. Moreover, in [1], it is shown that $\sharp A_n = p^{(ln+mp^n+O(1))p^{(r-1)n}}$ for any sufficiently large n.

In [3, 4, 5], a sufficient condition for the conjecture is given by using cup products of cyclotomic units of the p-cyclotomic field $k = \mathbb{Q}(\zeta_p)$. Assume that the Kummer-Vandiver conjecture holds for p, i.e., $A_0^+ = \dfrac{1+J}{2}A_0$ is trivial, where J is the complex conjugate. Note that A_0^+ is isomorphic to the p-part of the ideal class group of the maximal real subfield k^+ of k. Put $E'_{k,p} = \mathcal{O}_k[1/p]^\times/(\mathcal{O}_k[1/p]^\times)^p$ and $\mu_p = \langle \zeta_p \rangle$. By the assumption on A_0^+, $E'_{k,p}$ is generated by cyclotomic p-units. We consider the following cup product:

$$H^1(G_{k,p}, \mu_p) \times H^1(G_{k,p}, \mu_p) \to H^2(G_{k,p}, \mu_p^{\otimes 2}),$$

where $G_{k,p}$ is the Galois group of the maximal extension of k unramified outside p. This product can be represented as the following pairing:

$$E'_{k,p} \times E'_{k,p} \to (A_k/pA_k) \otimes \mu_p.$$

Let $\Delta = \mathrm{Gal}(k/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$, and let ω be the Teichmüller character. Put $J_p = \mathbb{Z}/(p-1)\mathbb{Z}$. For $j \in J_p$, denote by $Y^{(j)} = e_{\omega^j}Y$ the ω^j-part of $\mathbb{Z}_p[\Delta]$-module Y, where $e_{\omega^j} = \dfrac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\omega^j(\delta)\delta^{-1} \in \mathbb{Z}_p[\Delta]$. By decompositions, we can define the following pairing:

$$\begin{array}{ccc} \displaystyle\bigoplus_{j\in 2J_p}\left\{(E'_{k,p})^{(j)} \times (E'_{k,p})^{(2-i-j)}\right\} & \to & (A_k/pA_k)^{(1-i)} \otimes \mu_p \\ (c_p^{(j)}, c_p^{(2-i-j)}) & \mapsto & e_{i,j} = \langle c_p^{(j)}, c_p^{(2-i-j)}\rangle_i, \end{array}$$

where $i \in 2J_p$ and $c^{(j)} = [(1-\zeta_p)^{(j)}] \in (E'_{k,p})^{(j)}$. Put $I_p = \{i \in 2J_p \mid A_k^{(1-i)} \neq \{0\}\}$ and $r_p = \sharp I_p$.

Theorem 1.1 (Sharifi [4, 5]). *For $s_p = \sharp\{j \mid e_{i,j} \neq 0 \;\text{ for any }\; i \in I_p\}$,*

$$\mathrm{ht}_{\tilde{\Lambda}}(\mathrm{Ann}_{\tilde{\Lambda}} X_\infty) \geq \frac{s_p}{r_p^2 - r_p + 1} + 1.$$

In particular, data on (r_p, s_p) implies Greenberg's generalized conjecture for $p < 1{,}000$.

The procedure to check the conjecture is as follows:
(i) Compute r_p from the generating function of Bernoulli numbers modulo p.
(ii) Check $e_{i,i_0} \neq 0$ for some i_0 by computation of an ideal of Hecke ring.
(iii) Compute s_p from relations among $e_{i,j}$'s.
(iv) Apply data on r_p and s_p to Theorem 1.1.

In [4, 5], (ii) (resp. (iii)) is computed for $p < 1{,}000$ (resp.$25{,}000$) in the p-cyclotomic field. A lower bound of s_p is obtained by the number of zeros: $z_{p,i} = \sharp\{j \in 2J_p \mid e_{i,j} = 0\}$ for $i \in I_p$. In this paper, following the computation, we study the distribution of the number of zeros in the $4p$-cyclotomic field under a conjecture.

Theorem 1.2. *Let $z'_{p,i}$, $z'_{4p,i}$ and $z'_{4p,\chi,i}$ be the half number of nontrivial zeros in the pairing defined in §3. For $p < 2^{16}$, under Conjecture 4.1 for these prime numbers, the distributions are given in Table 1:*

Table 1. The number of (p, i) with $z' = m$.

m	0	1	2	3	4	5
$z'_{p,i}$	2523	640	80	4	0	0
$z'_{4p,i}$	2515	656	69	7	0	0
$z'_{4p,\chi,i}$	2013	973	241	50	5	1

Each distribution is similar to the Poisson distribution $Po(1/4)$ or $Po(1/2)$. We also obtain rare zeros which do not appear in the p-cyclotomic field.

2 Definition of maps and K_2-groups

We recall some definitions and theorems (cf. [3, §3]). The Milnor K_2-group of a commutative ring R is defined as follows:

$$K_2^M(R) = (R^\times \otimes R^\times)/\langle a \otimes (1-a);\ a,\, 1-a \in R^\times\rangle.$$

Let K be a number field with $K \supset \mu_p$. In [6], a particular choice of isomorphisms is described as a Chern class map:

$$ch_p :\ K_2(O_K[1/p])/p \tilde{\to} H^2(G_{K,p}, \mu_p^{\otimes 2}).$$

By a classical result of Matsumoto, we may identify $K_2^M(K)$ with $K_2(K)$. The group $K_2(O_K[1/p])$ may defined via the exact localization sequence:

$$0 \to K_2(O_K[1/p]) \to K_2(K) \to \bigoplus_{\mathfrak{q} \nmid p} k_\mathfrak{q}^\times \to 0,$$

where $k_\mathfrak{q}$ denotes the residue field of K at $\mathfrak{q}$. Since two p-units pair trivially under the tame symbol, the image of

$$K_2^M(O_K[1/p]) \to K_2^M(K) = K_2(K)$$

is contained in $K_2(O_K[1/p])$. This yields $K_2^M(O_K[1/p]) \to K_2(O_K[1/p])$ and

$$\kappa_p' :\ K_2^M(O_K[1/p])/p \to K_2(O_K[1/p])/p.$$

Here, the map

$$u_p : K_2^M(O_K[1/p])/p \to H^2(G_{K,p}, \mu_p^{\otimes 2})$$

coincides with $(-ch_p) \circ \kappa_p'$. Further, the natural map

$$n_p :\ E_{K,p}' \times E_{K,p}' \to K_2^M(O_K[1/p])/p$$

is surjective. Finally, define the map

$$\kappa_p :\ E_{K,p}' \times E_{K,p}' \to H^2(G_{K,p}, \mu_p^{\otimes 2}),$$

by $\kappa_p = u_p \circ n_p = (-ch_p) \circ \kappa_p' \circ n_p$.

Conjecture 2.1 (McCallum-Sharifi). *For all p and $k = \mathbb{Q}(\zeta_p)$ which satisfy the Kummer-Vandiver conjecture, κ_p' is surjective, i.e., by the Δ-decomposition,*

$$\kappa_{p,i} :\ (E_{k,p}' \times E_{k,p}')^{(2-i)} \to H^2(G_{k,p}, \mu_p^{\otimes 2})^{(2-i)},$$

is surjective for all $i \in J_p$.

3 Relations of a pairing in the $4p$-cyclotomic fields

Put $\zeta_{4p} = \zeta_4\zeta_p = \sqrt{-1}\zeta_p$, $K = \mathbb{Q}(\zeta_{4p}) = \mathbb{Q}(\sqrt{-1}, \zeta_p)$, $k = \mathbb{Q}(\zeta_p)$ $\tilde{\Delta} = \mathrm{Gal}(K/\mathbb{Q})$ and $\Delta = \mathrm{Gal}(k/\mathbb{Q}) \simeq \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-1}))$. We consider Δ as the subgroup of $\tilde{\Delta}$ by this isomorphism. The Dirichlet character group of $\tilde{\Delta}$ is $\{\chi^i\omega^j \mid i = 0, 1,\ j \in I_p\}$, where $\chi = \chi_{-4}$ is the Dirichlet character associated to $\mathbb{Q}(\sqrt{-1})$. Let J be the complex conjugate in Δ. We write $A^{\pm} = \dfrac{1 \pm J}{2}A$, $A^{(\chi^i,j)} = \tilde{e}_{\chi^i\omega^j}A$ for a $\mathbf{Z}_p[\tilde{\Delta}]$-module A, and $A^{(j)} = e_{\omega^j}A$ for a $\mathbf{Z}_p[\Delta]$-module A, where $\tilde{e}_{\chi^i\omega^j} = \dfrac{1}{\sharp\tilde{\Delta}}\sum_{\delta\in\tilde{\Delta}}\chi^i\omega^j(\delta)\delta^{-1} \in \mathbb{Z}_p[\tilde{\Delta}]$. Note that $\tilde{e}_{\chi^0\omega^j} = \dfrac{1+\tau}{2}e_{\omega^j}$, where $\langle\tau\rangle = \mathrm{Gal}(K/k)$. We also write $\alpha^{(\chi^i,j)}$ and $\alpha^{(j)}$ for an element of $\alpha \in A$.

Then, we have $I_p = I_{4p,\chi^0} = \{i \in 2J_p \mid A_k^{(1-i)} \simeq A_K^{(\chi^0,1-i)} \neq \{0\}\}$ and $r_p = \sharp I_{4p,\chi^0}$. Put $I_{4p,\chi} = \{i \in J_p \setminus 2J_p \mid A_K^{(\chi,1-i)} \neq \{0\}\}$ and $r_{4p,\chi} = \sharp I_{4p,\chi}$.

Even when p splits in $\mathbb{Q}(\sqrt{-1})$, the p-part of the subgroup generated by the ideal classes of prime ideals above p is also trivial in A_K. Hence, as in [3, §2], we have the following exact sequence:

$$0 \to (A_K/pA_K) \otimes \mu_p \to H^2(G_{K,p}, \mu_p^{\otimes 2}) \to \bigoplus_{v|p}\mu_p \to \mu_p \to 0.$$

Since $(A_K/pA_K)^{(\chi,0)}$ is trivial, this sequence implies that

$$(A_K/pA_K)^{(\chi,1-i)} \otimes \mu_p \simeq H^2(G_{K,p}, \mu_p^{\otimes 2})^{(\chi,2-i)}$$

for any $i \in I_{4p,\chi}$. We also have

$$(A_k/pA_k)^{(1-i)} \otimes \mu_p \simeq H^2(G_{k,p}, \mu_p^{\otimes 2})^{(2-i)}$$

for any $i \in I_p$.

In §3.1 and §3.2, we consider a pairing whose image is contained in $(A_K/pA_K)^{(\chi^0,1-i)} \otimes \mu_p \simeq (A_k/pA_k)^{(1-i)} \otimes \mu_p$ for $i \in I_p$. In §3.3, we consider a pairing which are contained in $(A_K/pA_K)^{(\chi,1-i)} \otimes \mu_p$ for $i \in I_{4p,\chi}$.

3.1 The p-cyclotomic field

We fix p and $i \in I_p \subset 2J_p$. For $j \in 2J_p$, put

$$j' = 2 - i - j \in 2J_p.$$

In §1–2, we introduce the following pairing:

$$\begin{array}{rcl} \kappa_{p,i} : \displaystyle\bigoplus_{j\in 2J_p}\left\{(E'_{k,p})^{(j)} \times (E'_{k,p})^{(j')}\right\} & \to & (A_k/pA_k)^{(1-i)} \otimes \mu_p \\ (c_p^{(j)}, c_p^{(j')}) & \mapsto & e_{i,j} = \langle c_p^{(j)}, c_p^{(j')}\rangle_i. \end{array}$$

Proposition 3.1. (cf. [3, §5]) *For all even integers a with $4 \leq a \leq p-1$,*

$$\sum_{j\in 2J_p}(1 + a^j - 2^j)(1 - 2^{j'})(1 - (a-1)^{j'})e_{i,j} = 0.$$

Further, for any $j \in 2J_p$,

$$e_{i,j} + e_{i,j'} = 0.$$

3.2 $4p$-cyclotomic field I

We fix p and $i \in I_p \subset 2J_p$. For $j \in I_p$, put

$$j' = 2 - i - j.$$

As in §2, we consider the following pairing:

$$\begin{array}{lll} \kappa_{4p,i} : \bigoplus_{j\in J_p\setminus 2J_p} \left\{(E'_{K,p})^{(\chi,j)} \times (E'_{K,p})^{(\chi,j')}\right\} & & \\ \quad \oplus \bigoplus_{j\in 2J_p} \left\{(E'_{K,p})^{(\chi^0,j)} \times (E'_{K,p})^{(\chi^0,j')}\right\} & \to & (A_K/pA_K)^{(\chi^0,1-i)} \otimes \mu_p \\ \quad (c_{4p}^{(\chi,j)}, c_{4p}^{(\chi,j')}) & \mapsto & f_{i,j} = \langle c_{4p}^{(\chi,j)}, c_{4p}^{(\chi,j')}\rangle_{4p,i}, \\ \quad (c_p^{(j)}, c_p^{(j')}) & \mapsto & e_{i,j} = \langle c_p^{(j)}, c_p^{(j')}\rangle_{4p,i}, \end{array}$$

where $c^{(\chi,j)} = [(1-\zeta_{4p})^{(\chi,j)}] \in (E'_{K,p})^{(\chi,j)}$. For $a \in \mathbb{Z}$ and $j \in J_p$, we define

$$u_{a,j} = \begin{cases} 1 & a \equiv 1 \bmod 4 \\ 2^j - 1 & a \equiv 3 \bmod 4. \end{cases}$$

Proposition 3.2. *For all odd integers a with $3 \le a \le p-2$,*

$$\sum_{j\in J_p\setminus 2J_p} (1-(-1)^{\frac{a-1}{2}}a^j) f_{i,j} + \sum_{j\in 2J_p} 2^{j-1}(2^j-1)(a^j-1)\left(2^{j'-1}(1-2^{j'}) + u_{a,j'}(a-1)^{j'}\right) e_{i,j} = 0.$$

Further, for any $j \in J_p \setminus 2J_p$,

$$f_{i,j} + f_{i,j'} = 0.$$

3.3 $4p$-cyclotomic field II

We fix p and $i \in I_{4p,\chi} \subset J_p \setminus 2J_p$. For $j \in J_p$, put

$$j' = 2 - i - j.$$

As in §2, we consider the following pairing:

$$\begin{array}{lll} \kappa_{4p,\chi,i} : \bigoplus_{j\in J_p\setminus 2J_p} \left\{(E'_{K,p})^{(\chi,j)} \times (E'_{K,p})^{(\chi^0,j')}\right. & & \\ \quad \left.\oplus (E'_{K,p})^{(\chi^0,j')} \times (E'_{K,p})^{(\chi,j)}\right\} & \to & (A_K/pA_K)^{(\chi,1-i)} \otimes \mu_p \\ \quad (c_{4p}^{(\chi,j)}, c_p^{(j')}) & \mapsto & g_{i,j} = \langle c_{4p}^{(\chi,j)}, c_p^{(j')}\rangle_{4p,\chi,i} \\ \quad (c_p^{(j')}, c_{4p}^{(\chi,j)}) & \mapsto & g_{i,j'} = \langle c_p^{(j')}, c_{4p}^{(\chi,j)}\rangle_{4p,\chi,i}, \end{array}$$

where $c^{(j')} \in (E'_{k,p})^{(j')} \simeq (E'_{K,p})^{(j')}$.

Proposition 3.3. *For all odd integers a with $3 \le a \le p-2$,*

$$\sum_{j\in J_p\setminus 2J_p} \{(1-(-1)^{\frac{a-1}{2}}a^j)\left(2^{j'-1}(1-2^{j'}) + u_{a,j'}(a-1)^{j'}\right) g_{i,j} + 2^{j'-1}(2^{j'}-1)(1-a^{j'}) g_{i,j'}\} = 0.$$

For all even integers a with $4 \le a \le p-1$,

$$\sum_{j\in J_p\setminus 2J_p} \{2^{j'-1}(2^{j'}-1) \quad ((a-1)^{j'}-1)g_{i,j} \\ +(1-(-1)^{\frac{a-2}{2}}(a-1)^j)\left(2^{j'-1}(1-2^{j'})+u_{a+1,j'}a^{j'}\right)g_{i,j'}\} = 0.$$

Further, for any $j \in J_p \setminus 2J_p$,

$$g_{i,j} + g_{i,j'} = 0.$$

3.4 Proofs of propositions

The anti-symmetry relation of each proposition is obtained form the anti-symmetry relation of $K_2(K)$. We prove the other relations by using special cyclotomic units. For $n,\ a \in \mathbb{Z}_{\ge 1}$, we define the following element of $\mathbb{Q}(\zeta_n)$:

$$\rho_{n,a} = \sum_{j=0}^{a-1}(-\zeta_n)^j = 1-\zeta_n+\zeta_n^2+\cdots+(-1)^{a-1}\zeta_n^{a-1}.$$

Then, we have

$$\rho_{n,a} = \frac{1+(-1)^{a-1}\zeta_n^a}{1+\zeta_n} = \begin{cases} \dfrac{1-\zeta_n^a}{1+\zeta_n} = \dfrac{(1-\zeta_n^a)(1-\zeta_n)}{1-\zeta_n^2} & a \equiv 0 \bmod 2 \\ \dfrac{1+\zeta_n^a}{1+\zeta_n} = \dfrac{(1-\zeta_n^{2a})(1-\zeta_n)}{(1-\zeta_n^a)(1-\zeta_n^2)} & a \equiv 1 \bmod 2. \end{cases} \tag{3.1}$$

For $n = 4p$, we have

$$1-\zeta_{4p}^a = \begin{cases} 1-\zeta_{4p}^a & a \equiv 1 \bmod 2 \\ \dfrac{1-\zeta_p^{2a}}{1-\zeta_p^a} & a \equiv 2 \bmod 4 \\ 1-\zeta_p^a & a \equiv 0 \bmod 4. \end{cases} \tag{3.2}$$

Proposition 3.4. (cf. [3, §5]) *For $n = p$ or $4p$, and $a \in \mathbb{Z}$ with $2 \le a \le p-1$, $\rho_{n,a}$ and $\rho_{n,a-1}$ are p-units in K. Further, the image $\kappa_p([\rho_{n,a}, \rho_{n,a-1}])$ is trivial in $H^2(G_{K,p}, \mu_p^{\otimes 2})$, where $[\rho_{n,a}, \rho_{n,a-1}]$ is the natural class of $(\rho_{n,a}, \rho_{n,a-1})$ in $E'_{K,p} \times E'_{K,p}$.*

Proof. Since $1-\zeta_p$ and $1-\zeta_{4p}$ are p-units in K, the first assertion follows immediately from the above expression. In the following, $\{x, y\}$ denotes an element in $K_2^M(K)/p$. For $n = p$, it is easy to see that $\{1-\zeta_n^a, \zeta_n\} = a^{-1}\{1-\zeta_n^a, \zeta_n^a\} = 0$ and $\{\rho_{n,a}, \zeta_n\} = 0$. For $n = 4p$, if $a \equiv 1 \bmod 2$, we have $\{1-\zeta_n^a, \zeta_n\} = a^{-1}\{1-\zeta_n^a, \zeta_n^a\} = 0$. If $a \equiv 0 \bmod 2$, as $\zeta_{4p} = \zeta_4\zeta_p$, we also have $\{1-\zeta_n^a, \zeta_n\} = \{1-\zeta_n^a, \zeta_p\} = 0$ by the above expression. These imply that $\{\rho_{n,a}, \zeta_n\} = 0$. Since $\zeta_n\rho_{n,a-1} = \zeta_n - \zeta_n^2 + \cdots + (-1)^{a-2}\zeta_n^{a-1} = 1-\rho_{n,a}$,

$$\{\rho_{n,a}, \rho_{n,a-1}\} = \{\rho_{n,a}, \zeta_n\} + \{\rho_{n,a}, \rho_{n,a-1}\} = \{\rho_{n,a}, 1-\rho_{n,a}\} = 0.$$

This implies the second assertion. □

Before the proofs of propositions, we give some equalities. For $j \in \mathbb{Z}$ with $(j, 4p) = 1$, let δ_j be an element of $\mathrm{Gal}(\mathbb{Q}(\zeta_{4p})/\mathbb{Q})$ satisfying $\zeta_{4p}^{\delta_j} = \zeta_{4p}^j$. Let $a \in \mathbb{Z}$ with $1 \le a \le p-1$. Then, there exists $a' \in \mathbb{Z}$ such that $a' \equiv 1 \bmod 4$ and $a' \equiv a \bmod p$. We have

$$[(1-\zeta_p^a)^{(\chi^0,j)}] = [(1-\zeta_p^a)^{\frac{1+\tau}{2}e_{\omega_j}}] = [(1-\zeta_p^a)^{(j)}] = [((1-\zeta_p)^{\delta_{a'}})^{(j)}] = (c^{(j)})^{\omega^j(\delta_{a'})} = (c^{(j)})^{a^j} \tag{3.3}$$

and

$$[(1-\zeta_p^a)^{(\chi,j)}] = [1]. \tag{3.4}$$

Further, if $a \equiv 1 \bmod 2$, we have

$$[(1-\zeta_{4p}^a)^{(\chi^0,j)}] = [(1-\zeta_{4p}^a)^{\frac{1+\tau}{2}e_{\omega_j}}] = \left[\left(\frac{1-\zeta_p^{4a}}{1-\zeta_p^{2a}}\right)^{(j)\frac{1}{2}}\right] = [(c^{(j)})^{2^{-1}(2a)^j(2^j-1)}] \tag{3.5}$$

and

$$[(1-\zeta_{4p}^a)^{(\chi,j)}] = [((1-\zeta_{4p})^{\delta_a})^{(\chi,j)}] = (c^{(\chi,j)})^{\chi\omega^j(\delta_a)} = (c^{(\chi,j)})^{\chi(a)a^j} = (c^{(\chi,j)})^{(-1)^{\frac{a-1}{2}}a^j}. \tag{3.6}$$

The first relation of Proposition 3.1. By Proposition 3.4, we have

$$\{\rho_{p,a}, \rho_{p,a-1}\} = \left\{\frac{(1-\zeta_p^a)(1-\zeta_p)}{1-\zeta_p^2}, \frac{(1-\zeta_p^{2(a-1)})(1-\zeta_p)}{(1-\zeta_p^{a-1})(1-\zeta_p^2)}\right\} = 0.$$

By (3.3), the contribution of $\rho_{p,a}^{(j)}$ to the coefficient is

$$a^j + 1 - 2^j,$$

and that of $\rho_{p,a-1}^{(j')}$ is

$$(2(a-1))^{j'} + 1 - (a-1)^{j'} - 2^{j'} = (1-2^{j'})(1-(a-1)^{j'}).$$

Therefore, we obtain the first relation.

The first relation of Proposition 3.2. By Proposition 3.4, we have

$$\{\rho_{4p,a}, \rho_{4p,a-1}\} = \left\{\frac{(1-\zeta_{4p}^{2a})(1-\zeta_{4p})}{(1-\zeta_{4p}^a)(1-\zeta_{4p}^2)}, \frac{(1-\zeta_{4p}^{a-1})(1-\zeta_{4p})}{1-\zeta_{4p}^2}\right\} = 0.$$

By (3.4) and (3.6), the contribution of $\rho_{4p,a}^{(\chi,j)}$ to the coefficient is

$$1 - \chi(a)a^j = 1 - (-1)^{(a-1)/2}a^j,$$

and that of $\rho_{4p,a-1}^{(\chi,j')}$ is 1. On the other hand, by (3.2), (3.3) and (3.5), the contribution of $\rho_{4p,a}^{(\chi^0,j)}$ is

$$(4a)^j - (2a)^j + 2^{-1}2^j(2^j-1) - 2^{-1}(2a)^j(2^j-1) - (4^j-2^j) = 2^{j-1}(2^j-1)(a^j-1),$$

and that of $\rho_{4p,a-1}^{(\chi^0,j')}$ is

$$\begin{cases} (a-1)^{j'} + 2^{-1}2^{j'}(2^{j'}-1) - (4^{j'}-2^{j'}) = 2^{j'-1}(1-2^{j'}) + (a-1)^{j'} & a \equiv 1 \bmod 4 \\ (2(a-1))^{j'} - (a-1)^{j'} + 2^{-1}2^{j'}(2^{j'}-1) - (4^{j'}-2^{j'}) \\ \qquad = 2^{j'-1}(1-2^{j'}) + (2^{j'}-1)(a-1)^{j'} & a \equiv 3 \bmod 4, \end{cases}$$

that is, $2^{j'-1}(1-2^{j'}) + u_{a,j'}(a-1)^{j'}$. Therefore, we obtain the first relation.

The first and second relations of Proposition 3.3. For an odd integer a, by Proposition 3.4, we have

$$\{\rho_{4p,a},\rho_{4p,a-1}\} = \left\{ \frac{(1-\zeta_{4p}^{2a})(1-\zeta_{4p})}{(1-\zeta_{4p}^{a})(1-\zeta_{4p}^{2})}, \frac{(1-\zeta_{4p}^{a-1})(1-\zeta_{4p})}{1-\zeta_{4p}^{2}} \right\} = 0.$$

By (3.4) and (3.6), the contribution of $\rho_{4p,a}^{(\chi,j)}$ to the coefficient is

$$1-\chi(a)a^{j} = 1-(-1)^{(a-1)/2}a^{j},$$

and that of $\rho_{4p,a-1}^{(\chi^0,j')}$ is

$$2^{j'-1}(1-2^{j'})+u_{a,j'}(a-1)^{j'}$$

as in Proposition 3.2.

On the other hand, by (3.2), (3.3) and (3.5), the contribution of $\rho_{4p,a}^{(\chi^0,j')}$ is

$$2^{j'-1}(2^{j'}-1)(a^{j'}-1),$$

and that of $\rho_{4p,a-1}^{(\chi,j)}$ is 1. Therefore, we obtain the first relation.

For an even integer a, by Proposition 3.4, we have

$$\{\rho_{4p,a},\rho_{4p,a-1}\} = \left\{ \frac{(1-\zeta_{4p}^{a})(1-\zeta_{4p})}{1-\zeta_{4p}^{2}}, \frac{(1-\zeta_{4p}^{2(a-1)})(1-\zeta_{4p})}{(1-\zeta_{4p}^{a-1})(1-\zeta_{4p}^{2})} \right\} = 0.$$

By (3.4) and (3.6), the contribution of $\rho_{4p,a}^{(\chi,j)}$ to the coefficient is 1, and that of $\rho_{4p,a-1}^{(\chi^0,j')}$ is

$$\begin{aligned}(4(a-1))^{j'}-(2(a-1))^{j'}+2^{-1}2^{j'}(2^{j'}-1)-(2^{-1}(2(a-1))^{j'}(2^{j'}-1))-(4^{j'}-2^{j'})\\ =2^{j'-1}(2^{j'}-1)((a-1)^{j'}-1).\end{aligned}$$

On the other hand, by (3.2), (3.3) and (3.5), the contribution of $\rho_{4p,a}^{(\chi^0,j')}$ is

$$\begin{cases} a^{j'}+2^{-1}2^{j'}(2^{j'}-1)-(4^{j'}-2^{j'}) & =2^{j'-1}(1-2^{j'})+a^{j'} \qquad a\equiv 0 \bmod 4\\ (2a)^{j'}-a^{j}+2^{-1}2^{j'}(2^{j'}-1)-(4^{j'}-2^{j'}) & =2^{j'-1}(1-2^{j'})+(2^{j'}-1)a^{j'} \qquad a\equiv 2 \bmod 4,\end{cases}$$

that is, $2^{j'-1}(1-2^{j'})+u_{a+1,j'}a^{j'}$. By (3.2), (3.4) and (3.6), that of $\rho_{4p,a-1}^{(\chi,j)}$ is

$$1-(-1)^{\frac{a-2}{2}}(a-1)^{j}.$$

Therefore, we obtain the second relation.

4 Proof of Theorem 1.2

Theorem 1.2 is obtained from numerical results under the following conjecture.

Conjecture 4.1. *$\kappa_{p,i}$ and $\kappa_{4p,i}$ (resp. $\kappa_{4p,\chi,i}$) are nontrivial for all p and $i\in I_p$, (resp. $I_{p,\chi}$).*

4.1 The p-cyclotomic field

Following computation in [5], we compute up to $p < 2^{16} = 65,536$.

Table 2. The distribution of p with $r_p = r$.

r	0	1	2	3	4	≥ 5
The number of p	3976	1979	497	86	4	0

Under Conjecture 4.1, we obtain the following table.

Table 3. The distribution of (p, i) with $z_{p,i} = m$.

m	2	3	4	5	6	7	8	9	10
$\sharp(p, i) \equiv (1, 0) \mod 4$	642	0	155	0	19	0	0	0	0
$\sharp(p, i) \equiv (1, 2) \mod 4$	0	0	597	0	165	0	19	0	2
$\sharp(p, i) \equiv (3, 0) \mod 4$	0	636	0	154	0	22	0	0	0
$\sharp(p, i) \equiv (3, 2) \mod 4$	0	648	0	166	0	20	0	2	0

Put $z_{p,i} = \sharp\{j \in 2J_p \mid e_{i,j} = 0\}$. First, we note that there are pairs of zeros by anti-symmetry $e_{i,j} = -e_{i,j'}$ when $j \neq j'$. In this paper, "index zeros" mean the pair of zeros which come from the index i, and "self zeros" mean zeros which come from the relation $\langle c, c\rangle = 0$. The other zeros are called "nontrivial zeros". We denote by $2z'_{p,i}$ the number of "nontrivial zeros" for p and i. By definition, we have

$$z_{p,i} = \sharp\{\text{nontrivial zeros}\} + \sharp\{\text{index zeros}\} + \sharp\{\text{self zeros}\}.$$

Since the number of index zeros is 2, we have

$$z_{p,i} = 2z'_{p,i} + 2 + \begin{cases} 0 & (p, i) \equiv (1, 0) \mod 4 \\ 2 & (p, i) \equiv (1, 2) \mod 4 \\ 1 & p \equiv 3 \mod 4. \end{cases}$$

The distribution of $z'_{p,i}$ is similar to the Poisson distribution $Po(1/4)$ as follows.

Table 4. The distribution of (p, i) with $z'_{p,i} = m$.

m	0	1	2	3
The number of (p, i)	2523	640	80	4
ratio	0.77702	0.19711	0.02464	0.00123
$Po(1/4)$	0.77880	0.19470	0.02434	0.00203

In the following examples, we write the ratio of $e_{i,j}/e_{i,0}$. There is no pair (p, i) with $e_{i,0} = 0$ in $p < 2^{16}$. We add the subscript j to zeros.

Example 4.1.
(1) $z_{101,68} = 4 = 2 + 2 + 0$ (nontrivial: 46-88, index: 66-68)
1, 84, 84, 89, 35, 29, 48, 15, 70, 31, 86, 53, 72, 66, 12, 17, 17, 100, 45, 61, 5, 75, 38, 0_{46}, 40, 20, 30, 66, 9, 28, 37, 95, 13, 0_{66}, 0_{68}, 88, 6, 64, 73, 92, 35, 71, 81, 61, 0_{88}, 63, 26, 96, 40, 56.
(2) $z_{379,100} = 3 = 0 + 2 + 1$ (index: 100-180, self: 140)
1, 97, …, 279, 159, 0_{100}, 258, …, 168, 0_{140}, 211, 173, …, 140, 121, 0_{180}, 220, …, 140, 206.
(3) $z_{379,174} = 3 = 0 + 2 + 1$ (index: 32-174, self: 292)
1, 310, …, 51, 0_{32}, 44, 143, …, 236, 335, 0_{174}, 328, 270, …, 325, 2, 0_{292}, 377, 54, …, 91, 63.

4.2 The $4p$-cyclotomic field I

Put $z_{4p,i} = \sharp\{j \in J_p \setminus 2J_p \mid f_{i,j} = 0\}$. Under Conjecture 4.1, we obtain the following tables.

Table 5. The distribution of of (p, i) with $z_{4p,i} = m$.

m	0	1	2	3	4	5	6	7	8
$\sharp(p,i) \equiv (1,0) \mod 4$	0	0	617	0	180	0	16	0	3
$\sharp(p,i) \equiv (1,2) \mod 4$	612	0	152	0	17	0	2	0	0
$\sharp(p,i) \equiv (3,0) \mod 4$	0	620	0	171	0	20	0	1	0
$\sharp(p,i) \equiv (3,2) \mod 4$	0	666	0	153	0	16	0	1	0

We can classify zeros into three types of zeros as in the p-cyclotomic case. Then, since the number of index zeros is 0, $z_{4p,i} = \sharp\{\text{nontrivial zeros}\} + \sharp\{\text{self zeros}\}$:

$$z_{4p,i} = 2z'_{4p,i} + \begin{cases} 2 & (p,i) \equiv (1,0) \mod 4 \\ 0 & (p,i) \equiv (1,2) \mod 4 \\ 1 & p \equiv 3 \mod 4. \end{cases}$$

The distribution of $z'_{4p,i}$ is also similar to the Poisson distribution $Po(1/4)$ as follows.

Table 6. The distribution of (p, i) with $z'_{4p,i} = m$.

m	0	1	2	3
The number of (p, i)	2515	656	69	7
ratio	0.77456	0.20203	0.02125	0.00216
$Po(1/4)$	0.77880	0.19470	0.02434	0.00203

In the following examples, we write the ratio of $f_{i,j}/e_{i,0}$. We add the subscript j to zeros.

Example 4.2.
(1) $z_{4\cdot379,100} = 3 = 2 + 1$ (nontrivial: 317-341, self: 329)
2, 98, ..., 137, 0_{317}, 212, 13, 262, 310, 227, 0_{329}, 152, 69, 117, 366, 167, 0_{341}, 242, ..., 45, 5.
(2) $z_{4\cdot379,174} = 3 = 2 + 1$ (nontrivial: 267-317, self: 103)
306, 29, ..., 193, 121, 0_{103}, 258, ..., 225, 0_{267}, 89, ..., 347, 290, 0_{317}, 154, 124, ..., 36, 141.
(3) $z_{4\cdot929,820} = 6 = 4 + 2$ (nontrivial: 1-109, 139-899, self: 55, 519)
0_1, 383, ..., 68, 0_{55}, 861, 670, 750, ..., 7, 546, 0_{110}, 110, 75, ..., 69, 394, 0_{139}, 272, 299, ..., 804, 829, 104, 461, 0_{519}, 468, 825, ..., 630, 657, 0_{899}, 535, 860, ..., 854, 819.

The zeros 0_{317} in (1) and (2) are very rare, because they come from the nontriviality of the p-part of the ideal class group of K^+. In other words, they come from the nontriviality of χ_{-4}-part of $K_{4m+2}(\mathbb{Z}[\sqrt{-1}])[p]$, where $p = 379$ is the unique prime number satisfying the nontriviality in $p < 20,000,000$ (cf. [7, 8, 9]).

The zeros 0_1 in (3) is rare, because there is only one pair $(p, i) = (929, 820)$ satisfying the condition in $p < 2^{16}$ and $i \in I_p$.

4.3 The $4p$-cyclotomic field II

Table 7. The distribution of p with $r_{4p,\chi} = r$.

r	0	1	2	3	4	5	≥ 6
The number of p	3960	1993	492	80	14	3	0

Put $z_{4p,\chi,i} = \sharp\{j \in J_p \mid g_{i,j} = 0\}$. Under Conjecture 4.1, we obtain the following tables except for (p, i) =(9511, 2221), (12073, 7547), (13367, 5331), (30241, 19981), (31649, 8903), for which the relations in Proposition 3.2 is clearly insufficient, because $2^{2-i} \equiv 1$ or $2 \bmod p$ (cf. [3, §5]).

Table 8. The distribution of of (p,i) with $z_{4p,\chi,i} = m$.

m	0	1	2	3	4	5	6	7	8	9	10
$\sharp(p,i) \equiv (1,1) \mod 4$	0	0	478	0	244	0	58	0	16	0	1
$\sharp(p,i) \equiv (1,3) \mod 4$	0	0	504	0	235	0	63	0	11	0	1
$\sharp(p,i) \equiv (3,1) \mod 4$	0	0	511	0	259	0	64	0	10	0	2
$\sharp(p,i) \equiv (3,3) \mod 4$	0	0	520	0	235	0	56	0	13	0	1

Here $z_{4p,\chi,i}$ is even, because $g_{i,j} = -g_{i,j'}$ with $j \not\equiv j' \bmod 2$. We can classify zeros into three types of zeros as in the p-cyclotomic case. Then, since the number of self zeros is 0, $z_{4p,\chi,i} = \sharp\{\text{nontrivial zeros}\} + \sharp\{\text{index zeros}\}$:

$$z_{4p,\chi,i} = 2z'_{p,\chi,i} + 2.$$

The distribution of $z'_{4p,\chi,i}$ is similar to the Poisson distribution $Po(1/2)$ as follows.

Table 9. The distribution of (p,i) with $z'_{4p,\chi,i} = m$.

m	0	1	2	3	4	5
The number of (p,i)	2013	973	241	50	5	1
ratio	0.61316	0.29638	0.07341	0.01523	0.00152	0.00030
$Po(1/2)$	0.60653	0.30327	0.07582	0.01264	0.00158	0.00016

In the following examples, we write the ratio of $g_{i,j}/g_{i,1}$ (resp. $g_{i,j}/g_{i,p-2}$) if $g_{i,1} \neq 0$ (resp. $g_{i,1} = 0$). We add the subscript j to zeros.

Example 4.3.
(1) $z_{4\cdot 379,\chi,317} = 2 = 0 + 2$ (index: 317-317')
1, 109, $\ldots$, 285, 369, 0_{317}, 331, 119, $\ldots$, 354, 222,
-1, -109, $\ldots$, -285, -369, $0_{317'}$, -331, -119, $\ldots$, -354, -222.
(2) $z_{4\cdot 941,\chi,687} = 2 = 0 + 2$ (nontrivial 1-1', index: 687-687')
0_1, 413, 589, 110, $\ldots$, 257, 437, 0_{687}, 314, 569, 300, 212, $\ldots$, 462, 331, 596, 13, 1,
$0_{1'}$, -413, -589, -110, $\ldots$, -257, -437, $0_{687'}$, -314, -569, -300, -212, $\ldots$, -462, -331, -596, -13, -1.

The zero 0_1 in (2) is rare, because there is only one pair $(p,i) = (941, 887)$ satisfying the condition in $p < 2^{16}$ and $i \in I_{4p,\chi}$. There is no zero 0_j with $j \equiv 2 - i \bmod (p-1)$ and $j' \equiv 0 \bmod (p-1)$ in the range.

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