Euler's mathematics and "Letters to a German Princess"

Mysterious numbers left by Euler

Hiroki Sumida-Takahashi
Department of Mathematics, Graduate School of Science
Hiroshima University

Leonhard Euler, the greatest mathematician of the 18th century, has a large number of numerical errors in his major work "Introduction to Infinite Analysis"(1748) (Japanese translation [4]), as pointed out in the complete collection [3]. [9] introduced the following hypothesis, stating that it is difficult to attribute these errors to simple calculation errors.

> **The subtle errors in the large number of numerical values in "Introduction to Infinite Analysis" are not mere mistakes, but are Euler's intentional "problems", and the solutions are written in "Letters to a Princess".**

In this essay, we summarize the reasons for the hypothesis and speculate about the intent of the "problems".

First, in §1, we present the numerical list and errors, and explain the abnormalities. Next, in §2, we point out phrases and figures that are considered to be hints for solving the "problems". In §3 , we provide proposed solutions to the "problems" and explain how to derive and confirm them. Finally, in §4, we infer the intent of the "problems" by referring to some of Euler's claims.

The "problems" are thought to be related to the various academic fields that Euler explored, and are assumed to be extremely important for understanding his behavioral principles. At present, only incomplete solutions to the "problems" have been presented, but we hope that the day will come when we can find widely understood solutions by the challenge of many researchers who admire Euler.

Furthermore, Euler's personality as a problemer is widely known, as he left behind many important sources over vast areas (see [2][5][6]). Our hypothesis emphasizes this once again, and does not contradict the conventional understanding on Euler.

§1. Numerical errors in "Introduction to Infinite Analysis"

The numerical errors are only near the last digit. The list below shows the differences between the numbers calculated in [7] and those in the "Introduction". For example, the correct truncated value for $\log_{10} 2$ with seven decimal places is 0.3010299, but the "Introduction" says it as 0.3010300 .

Volume 1

Chapter 6, p.77

$\log_{10} 2$	=	0.3010299	+1
$1/\log_{10} 2$	=	3.3219280	−3

Chapter 7

P1	$\log n$	
$l7$	1.94591014905531330510535 27	+1112
$l8$	2.07944154167983592825169 63	+1
$l9$	2.19722457733621938279049 04	+1

Chapter 8

P2	$s_k = (-1)^{\frac{k-1}{2}} \dfrac{\pi^k}{2^k k!}$			$c_k = (-1)^{\frac{k}{2}} \dfrac{\pi^k}{2^k k!}$	
01	$+1.57079632679489661923132169 16$	-0	00	$+1.00000000000000000000000000$	-0
03	$-0.64596409750624625365575656 38$	$+2$	02	$-1.23370055013616982735431137 49$	$+4$
05	$+0.07969262624616704512050554 94$	-6	04	$+0.25366950790104801363656336 63$	-4
07	$-0.00468175413531868810068546 39$	$+7$	06	$-0.02086348076335296087305163 72$	$+8$
09	$+0.00016044118478735982187266 08$	-3	08	$+0.00091926027483942658024171 62$	-4
11	$-0.00000359884323521208534045 85$	$+5$	10	$-0.00002520204237306060548105 30$	$+4$
13	$+0.00000056921729219679268117 7$	-6	12	$+0.00000047108747788181715036 70$	-5
15	$-0.00000000066880351098114672 32$	$+8$	14	$-0.00000000638660308379185224 10$	$+2$
17	$+0.00000000000606693573110619 56$	-6	16	$+0.00000000006565963114979472 36$	-6
19	$-0.00000000000004377065467313 74$	$+4$	18	$-0.00000000000052944002007346 23$	$+3$
21	$+0.00000000000000025714228928 60$	-4	20	$+0.00000000000000343773917909 86$	-5
23	$-0.00000000000000000125389954 05$	$+2$	22	$-0.00000000000000001835991652 15$	$+3$
25	$+0.00000000000000000000515645 51$	-1	24	$+0.00000000000000000008206753 30$	-3
27	$-0.00000000000000000000001812 39$	$+0$	26	$-0.00000000000000000000031152 84$	-1
29	$+0.00000000000000000000000005 50$	-1	28	$+0.00000000000000000000000101 67$	-2
			30	$-0.00000000000000000000000000 28$	$+2$

P3	$\dfrac{2(2^{n+1}-1)\pi^n \lvert B_{n+1}\rvert}{(n+1)!} - \dfrac{4}{\pi}$		$\dfrac{1}{2^{n-1}\pi} - \dfrac{2\pi^n \lvert B_{n+1}\rvert}{(n+1)!}$	
05	$+0.0018424752035$	-1	-0.0065510747882	
07	$+0.0001975800715$	-1	-0.0003450292553	-1
09	$+0.0000216977373$	-128	-0.0000202791060	
11	$+0.0000024011369$	$+1$	-0.0000012366527	
13	$+0.0000002664133$	-1	-0.0000000764958	-1

Chapter 11

PA	$\zeta(k)\left(1-\dfrac{1}{2^k}\right)$		**PB**	$\zeta(k)\dfrac{1}{2^k}$	
E	$= 1.00001704136304482548818$	$+1998$	ε	$= 0.00097753376477325984896$	$+2$
F	$= 1.00000188584858311957590$		ξ	$= 0.00024420070472492872273$	$+1$
H	$= 1.00000002323715737915670$		θ	$= 0.00001525902225127271503$	-1526
L	$= 1.00000000003186677514044$		λ	$= 0.00000023841863595259255$	-101
Q	$= 1.00000000000000053965957$		π	$= 0.00000000023283064370808$	-1
W	$= 1.00000000000000000000913$		ϕ	$= 0.00000000000022737367545$	-1
			ω	$= 0.00000000000000355271368$	-1

Chpater 15

PC	$\upsilon(k) = \sum_{p:prime} \frac{1}{p^k}$	
02	0.452247420041065	+157
04	0.076993139764246	+6
06	0.017070086850637	+2
08	0.004061405366518	−3
10	0.000993603574437	−804
12	0.000246026470035	−2
20	0.000000953961124	−1
30	0.000000000931326	−3

Volume 2

Chapter 21	$(\sim \cos\log 2)$		Chapter 22	$(\sin 1 \sim)$	
$\log_{10} 2^{\sqrt{2}}$	$= 0.4257207$	+67	$\cos\log 2$	$= 0.76923890136397$	−989
$2^{\sqrt{2}}$	$= 2.665144$	+42	$\sin 1$	$= 0.84147098480789$	−275
$10^{\sqrt{2}}$	$= 25.954553$	+1317	$\cos 1$	$= 0.54030230586813$	−2472

$s + \cot s$	
−0.17200818	−1
−0.09062597	−1
−0.05892836	−2
−0.04258548	−5

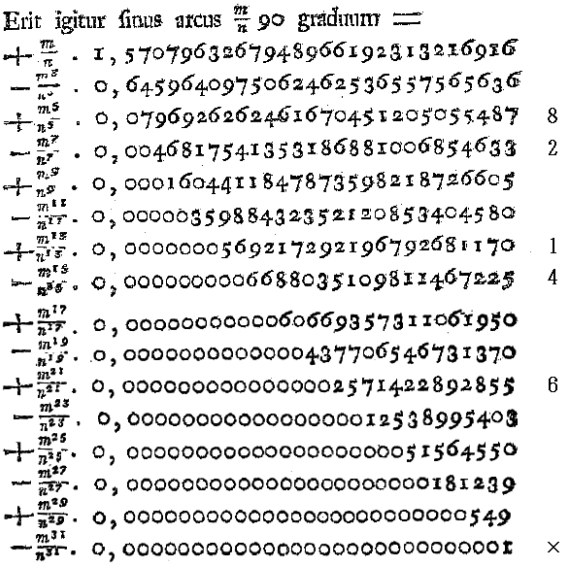
In particular, the rate of errors in the lists of sine and cosine is high. We point out the following abnormalities in [9].

1. There are errors in 28 out of 31 numbers.
2. The accuracy is mysterious, with 28 digits after the decimal point.
3. The ratios of errors are rapidly increasing.
4. The errors are within the last digit.
5. The absolute value of only one data is greater than the positive value.
6. **If we make the errors correspond to musical scales, we can obtain an extremely skillfully composed score.**

Naturally, if we can explain the reason of errors by a simple algorithm, we do not need to consider hypothesis such as Euler's intentions. However, more than three years have passed since we presented this issue, and no such explanation has been obtained.

Even more troubling, a list on the next page that is almost identical to the list in P2 was published in E128 (written in 1739, published in 1750), and the errors in this list must also be considered. Most of the numbers are the same, but the last digit of the sine coefficient has changed from 825014006 to 735005005. None of the numbers that changed have been corrected, and the number of errors remains at 28.

In the end, it must be said that it is probably impossible to systematically explain the errors in these two lists from a single algorithm.



Maclaurin expansion of the sine of E128

§2. Hints in "Introduction to Infinite Analysis", "Letters to a Princess", and "Beautiful Relationships"

Against the hypothesis, there should be a reasonable objection that it is difficult to express concrete intentions using abstract numbers. This is because numbers can represent a variety of objects, allowing for a variety of interpretations. Therefore, if they are fair riddles, there should be hints that direct the interpretations of the numbers.

In the following, we list main figures and phrases which we consider to be hints. Due to space limitations, please see [7] and [8] for details. Furthermore, we would like to apologize to the problemer and potential solvers for the unscrupulous act of specifying hints on problems.

"Introduction to Infinite Analysis" (written in 1745, published in 1748) Diagram at the beginning of Volume 1

Fig.1

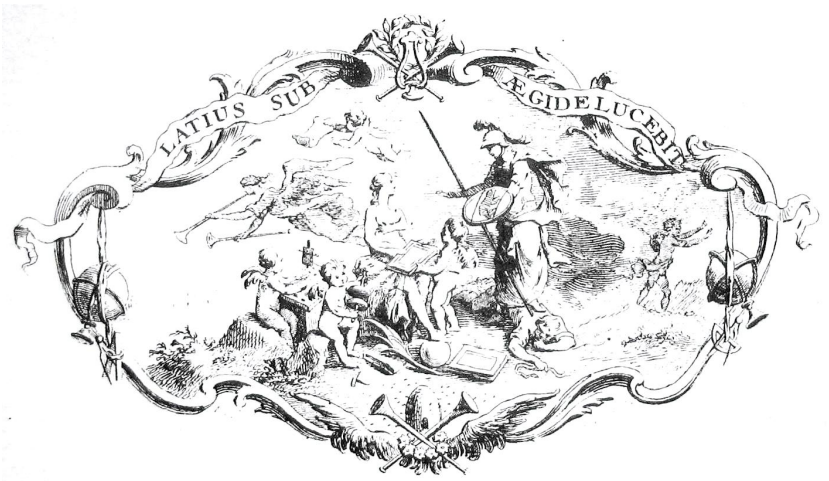


Fig.2

Fig.3

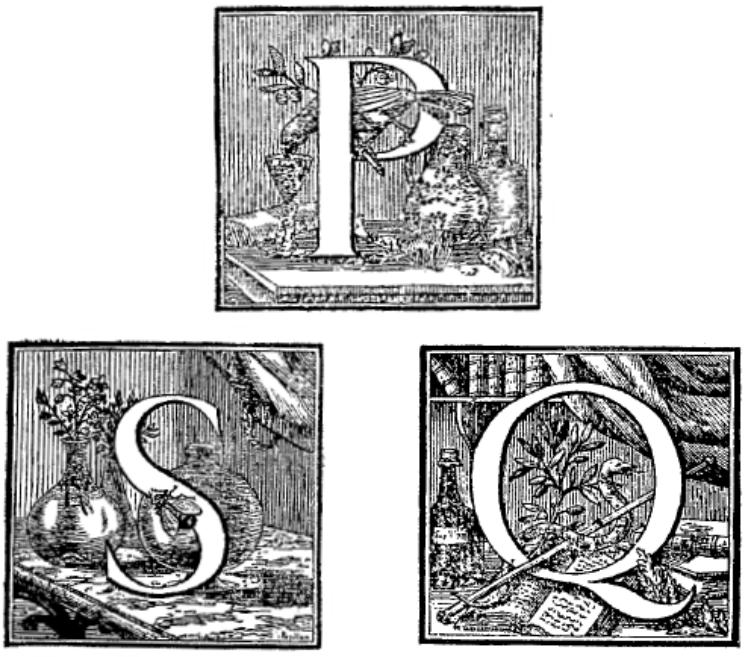


Fig.4

Examples from Volume 1, Chapter 6

Example B $\log_{10} 5$ (the numbers on the right are the correct values)

$A = 1.000000$		$lA = 0.0000000$		
$B = 10.00000$		$lB = 1.0000000$		$C = \sqrt{AB}$
$C = 3.162277$		$lC = 0.5000000$		$D = \sqrt{BC}$
$D = 5.623413$		$lD = 0.7500000$		$E = \sqrt{CD}$
$E = 4.216964$	5	$lE = 0.6250000$		$F = \sqrt{DE}$
$F = 4.869674$	5	$lF = 0.6875000$		$G = \sqrt{DF}$
$G = 5.232991$		$lG = 0.7187500$		$H = \sqrt{FG}$
$H = 5.048065$		$lH = 0.7031250$		$I = \sqrt{FH}$
$I = 4.958069$	8	$lI = 0.6953125$		$K = \sqrt{HI}$
$K = 5.002865$	4	$lK = 0.6992187$		$L = \sqrt{IK}$
$L = 4.980416$		$lL = 0.6972656$		$M = \sqrt{KL}$
$M = 4.991627$		$lM = 0.6982421$		$N = \sqrt{KM}$
$N = 4.997242$	3	$lN = 0.6987304$		$O = \sqrt{KN}$
$O = 5.000052$	3	$lO = 0.6989745$	6	$P = \sqrt{NO}$
$P = 4.998647$		$lP = 0.6989525$		$Q = \sqrt{OP}$
$Q = 4.999350$		$lQ = 0.6989135$		$R = \sqrt{OQ}$
$R = 4.999701$		$lR = 0.6989440$		$S = \sqrt{OR}$
$S = 4.999876$	7	$lS = 0.6989592$	3	$T = \sqrt{OS}$
$T = 4.999963$	5	$lT = 0.6989668$	9	$V = \sqrt{OT}$
$V = 5.000008$	9	$lV = 0.6989707$		$W = \sqrt{TV}$
$W = 4.999984$	7	$lW = 0.6989687$	8	$X = \sqrt{WV}$
$X = 4.999997$	8	$lX = 0.6989697$	8	$Y = \sqrt{VX}$
$Y = 5.000003$		$lY = 0.6989702$	3	$Z = \sqrt{XY}$
$Z = 5.000000$		$lZ = 0.6989700$		

Example C1 Find the value of $2^{\frac{7}{12}}$.

Example C2 When the population of a certain area increases by 1/30 every year, assuming that 100,000 people initially lived there, find the population after 100 years.

Example C3 What rate does the population increase each year, assuming that the number of people was 6 after the flood, and that 200 years later the population reaches 1,000,000?

Example D1 How many years will it take for the population to increase 10 times, assuming that the population increases by 1/100?

Volume 2 Chapter 22 Problems (Problema) and Solutions (Solutio)

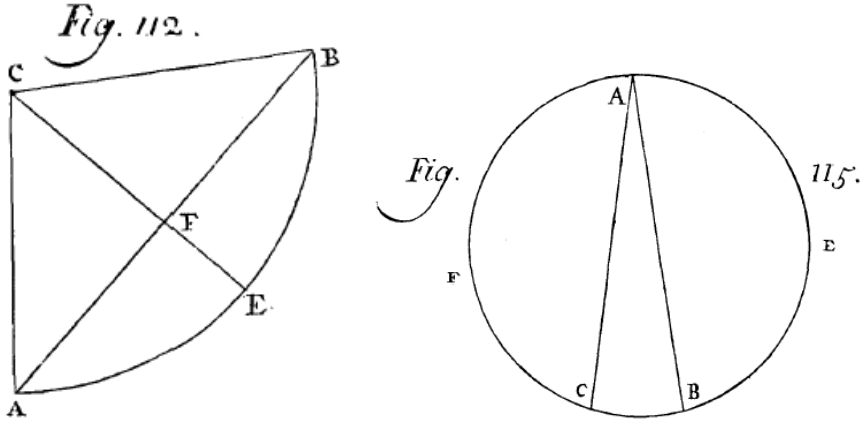


Fig.5

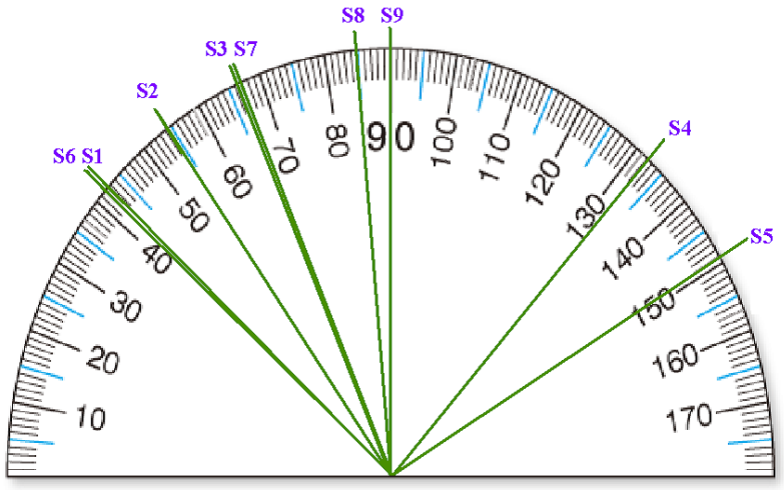
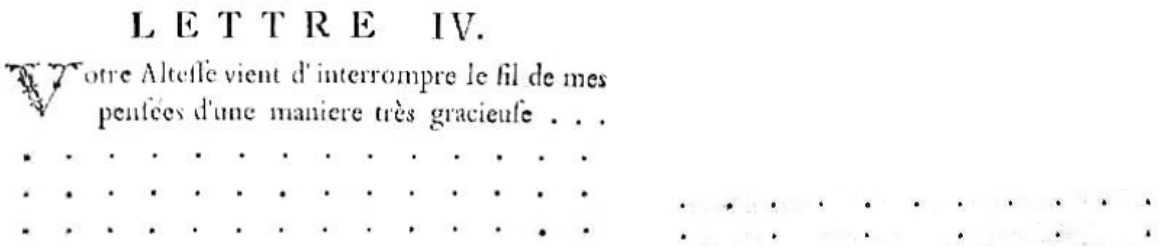


Fig.6

"Letters to a Princess" (written in 1760-62, published in 1768-70)
First letter

"It is all the work of the Almighty that is vast and boundless".

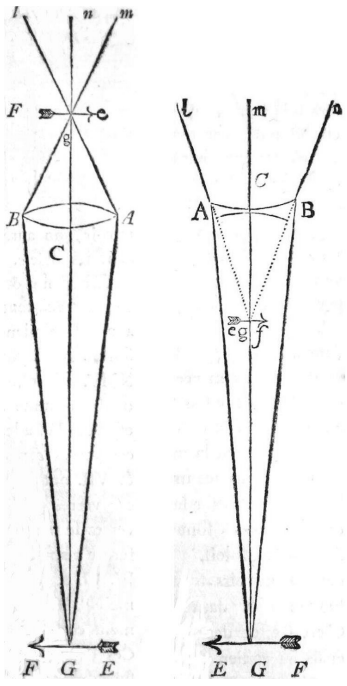
4th letter



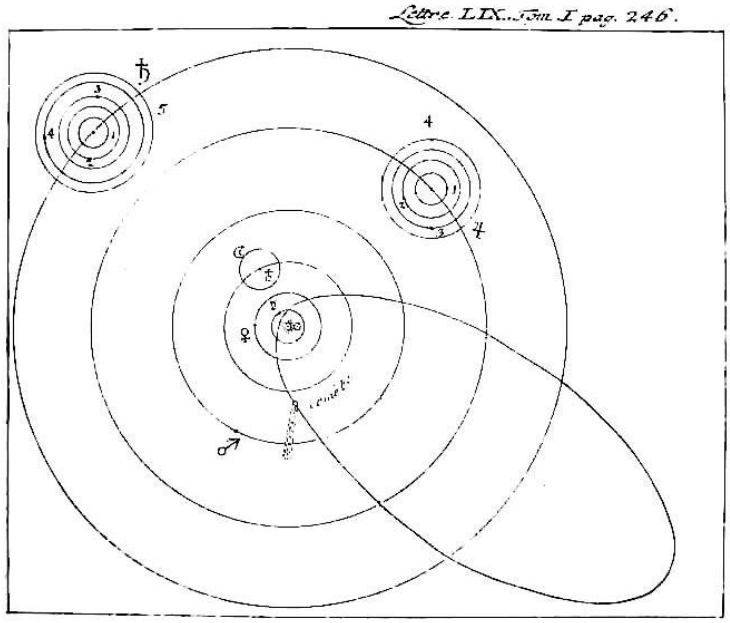
LETTRE IV.

Votre Alteſſe vient d'interrompre le fil de mes penſées d'une maniere très gracieuſe . . .

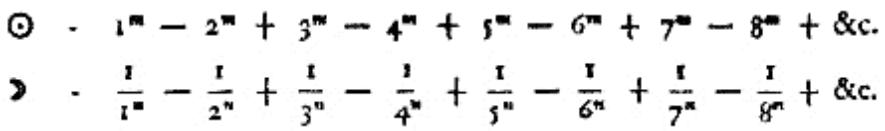
39th letter



59th letter



"Beautiful Relations"(written in 1749, published in 1768)



§3. Suggested solutions to "problems"

First, we would like to note that the parts that are interpreted as problems, hints, and solutions have little impact on the main theory of each work. In other words, even if the approximations, examples, and diagrams are slightly wrong or strange, they probably will not cause any real harm to anyone.

Hints for "Introduction to Infinite"

Fig.1 shows "Algebra (algebraic calculations)", "Analysis (sine and differential calculations)", "Geometry (determined trees and protractors)", and "Entry to infinitesimal analysis" is depicted. This is clearly stated in the introduction as a goal of "Introduction to Infinite Analysis".

In Fig.2, "music (harp, trumpet)" and "geometry (protractor, compass, globe)" are depicted. Sound and light have been used since ancient times as a means of communication between distant places. It is thought to be a drawing related to Athena, the goddess of wisdom, art, crafts, and strategy in Greek mythology, and was used in Bousquet's publications.

In Fig.3, "anatomy (birds, mammals, humans)", "geometry (compass, diagram)", and "astronomy (telescope)" are depicted. A person can be seen writing, but the content is unknown. Anatomy seems a little abrupt here because the relationship with mathematics is not clear. On the other hand, in the Pythagorean school of ancient Greece, the etymology of the word Mathematics, "mathematica" (meaning subjects to be learned), was the four branches of "arithmetic, music, geometry, and astronomy", so there is no sense of discord in their appearance.

In Fig.4, P stands for "pigeon and olive", S stands for "a small animal like a fly or bee", and Q stands for "snake". P reminds us of the dove that brought the olives to Noah's ark in Genesis in the Old Testament, and Q reminds us of Moses transforming into a serpent in the book of Exodus. Because of this association, it would be natural to think of S's small animal as a "bee". This is because when God gave Moses a serpent, he searched for the problem of where the dove brought the olives, and in the end he found Jerusalem, the promised land flowing with milk and honey" was the solution . Recognizing that this type of problem is found in "Introduction to Infinite Analysis" will make it easier to understand Euler's "problems".

If we read the last number in the list of numbers on the left in Example B , reading it up and down alternately, it becomes (73-37,44,1)-(59,48,5)-(67,36,2)-(157,2), if we read the last number from the bottom of the number list on the right, it will be 20-777-028-555-7614. Although it is a meaningless mistake at this point, it is presumed that all solutions will be checked in the end.

$2^{\frac{7}{12}}$ in Example C1 is a famous number as the frequency ratio of a perfect fifth in equal temperament in "music". The flood in Example C3 is Noah's flood in the book of Genesis in the Old Testament, and the six people are presumably Noah's sons and their wives. This assumption is not difficult if we realize that the letter pictures of P, S, and Q are quotations from the Old Testament. In the calculation of example C2 , the number $31/30 = 1.033333\cdots$ appears, and in the calculation of example D1, the number $101/100 = 1.01$ appears.

Fig. 5 shows two of the diagrams for problems in Chapter 22 of Volume 2 where D is not written for some reason. There are a total of 9 problems to find angles, and QEI and QEF appear at the end of the solution, but please note that the famous QED does not appear. The number of vertices of these two graphs is 5 . Then, while the dual graph in Fig.112 remains unchanged, the dual graph in Fig.115 changes to the seven bridges of Königsberg .

As shown in Fig.6, solutions S1 and S6 approximate approximately 42 degrees, S2 approximates 54 degrees, S3 and S7 approximate 67 degrees, S4 approximates 132 degrees, S5 approximates 149 degrees, and S8 approximates 84 degrees, and the first solution for S9 is written as

90 degrees - 90 degrees. Here, note that 90 degrees$= 1.57079\cdots$.

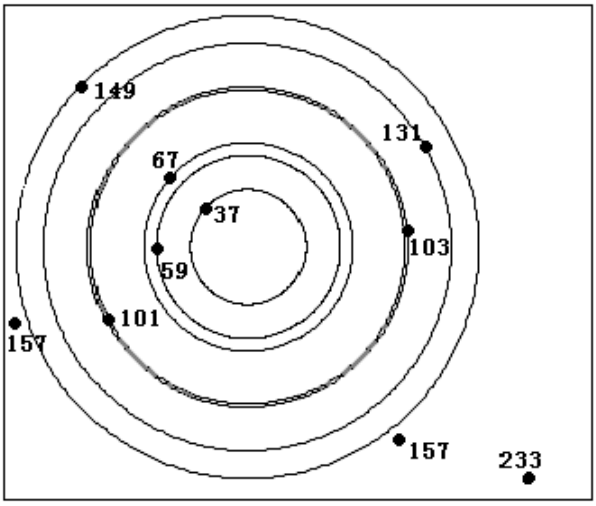
Draft solutions of "problems" in "Introduction to Infinite Analysis"

P1: The first two capital letters in the sentence that caused P1's error are "PS". Since "music" has already appeared in example C1, and "Old Testament" has appeared in example C3, Psalms comes to mind ([7] Chapters 4 and 10). Psalm 111, verse 2 , says, "Great is the work of God; everyone who loves it seeks it." Euler was a Protestant and an explorer of the world.

P2: Since psalms are songs, there should be a musical scores. By correlating the error in P2 to the scale, a song with 8686 common meters is composed, and the "music" shown in the opening diagram appears. (For sheet music, see [8] Appendix C or [9] . The basic beat is 42+6+6=54 in "Hymn" (The Board of Publications The United Church of Christ in Japan) 10, 11)

P3: Using the "geometry" shown in the opening diagram as a hint, the seven bridges of the Königsberg come to mind from the positions of the seven errors in P3 ([7] Chapter 6). Note that $128 = 2^7$.

PA, PB, PC: When the large errors in PA and PC are factorized, the smallest irregular prime number 37 and the third 67 appear from PA:$1998 = 54 \times 37$, PC:$804 = 12 \times 67$. An irregular prime number is a prime number that appears in the numerator of the rational part of the zeta value. The combined errors of PA and PB are $472 = 8 \times 59$, and the second irregular prime number 59 also appears. (For the reason for addition, see Part 2 , Chapters 5 and 11 of [8]) Below are the irregular prime numbers and exponents up to the 9th.
$(37, 32)$ $(59, 44)$ $(67, 58)$ $(101, 68)$ $(103, 24)$ $(131, 22)$ $(149, 130)$ $(157, 62)$ $(157, 110)$ $(233, 84)$



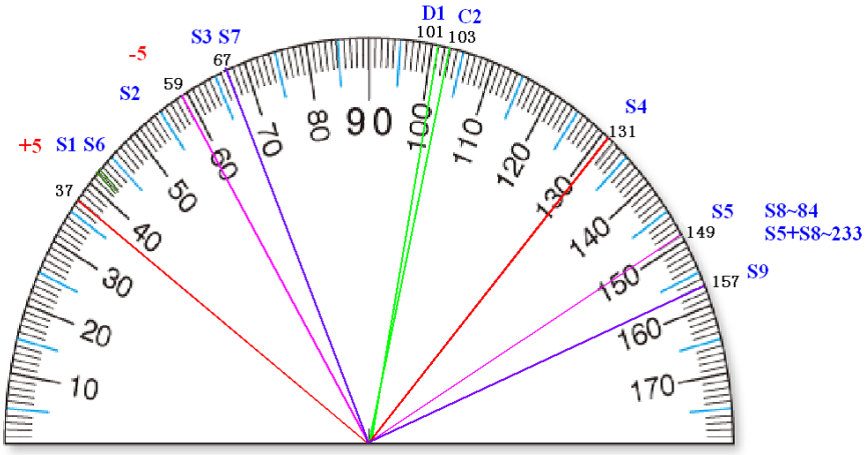
Irregular primes appear in the numerator of the zeta value with a period of $p - 1$. If we take this periodicity into account and plot pairs of irregular primes and exponents clockwise, we get the diagram above. This diagram reminds us of the solar system, and the zeta value gives rise to the term "astronomy". The relationship between zeta and the solar system is a unique idea, but since zeta is represented by the symbols of the sun and moon in "Beautiful Relationships", it is easier to make the connection by referring to this essay.

Confirmation of solutions in "Introduction to Infinite Analysis" Volume 2

PC, PB, PA : Confirmation of the musical score in which three irregular prime numbers from 67, 42, 1317 are expressed. ([8] Appendix C)

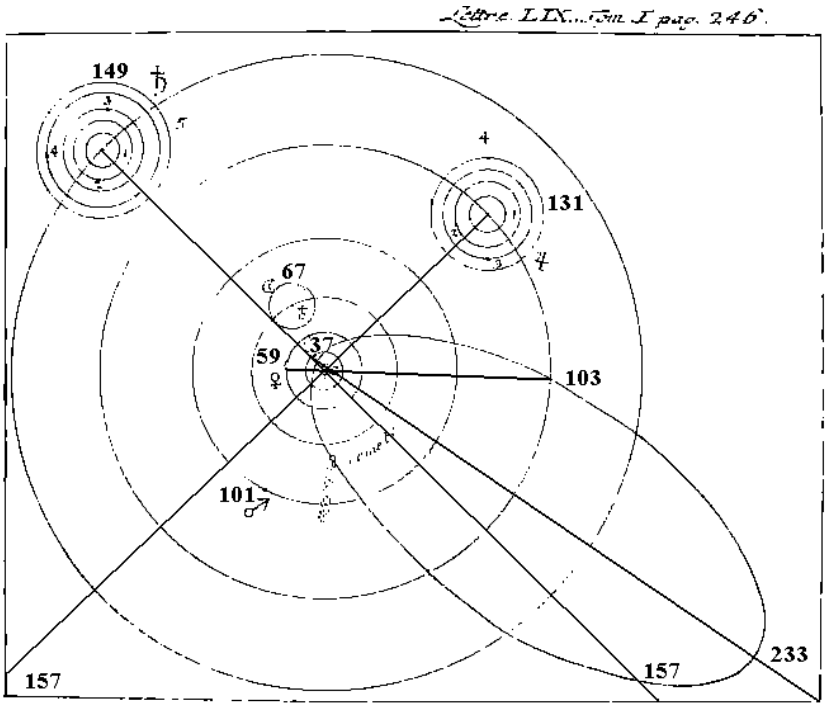
P3, P2, P1 : Check the sum of errors from "-989, -275, -2472". Check the solution based on the error. ([7] Chapter 8)

Graph of P3 : Confirmed from the two graphs in Fig. 5 and their dual graph. ([7] Chapter 6)



Irregular primes : C2≈1.03, D1=1.01, S3≈ S7≈67, S4≈131, S5≈149, S5+S8≈233, S9= 1.57 − 1.57.
S1≈S6≈42∼37, the diffrence is +5. S2≈54∼59, the difference is −5.

Confirmation of solutions in "Letters to a Princess"
P1: Confirmation of the first half of PS.111-2 from the last words of the first letter. The second half is the 20th letter.
P2: Check the music score and psalm from the dots in the 4th letter.(42+11+6=59 beats, 11-12, etc.)
P3: Divide the graph on the left where D is not written in the 39th letter vertically into Fig. 112, and divide it horizontally into Fig.115, Confirmation of the seven bridges in Königsberg from the graph on the right.
PA, PB, PC: You can confirm the diagram on the previous page by solar system diagram in the 59th letter. If the ratios of the widths of orbits are 16 : 22 : 37 : 67 : 103 : 101 : 131 : 149, the planets and irregular prime numbers almost correspond (see [7] p.183). The symbols of Jupiter and Saturn are 4 and 5, which correspond to 131 and 149, which correspond to S4 and S5 in "Introduction".



The above suggested solutions can be summarized as follows, using "Letters to the Princess" as a reference.

7 Lists	functions	academic areas	physics	contents
B	$\log_{10} 5$	Arithmetic	number	all solutions
P1	log	Theology	space	lyric for a psalm
P2	$\sin, \cos$	Music	sound	score for a psalm
P3	$\tan, \cot$	Geometry	light	graph for a psalm
PA	ζ A	Astronomy	gravity	37
PB	ζ B	Astronomy	gravity	59
PC	ζ C	Astronomy	gravity	67

§4. Intent of "problems"

Euler's claims are listed below.

· About the fountain of discovery in mathematics (from the introduction of "Introduction")
"Indeed, I would like to state without hesitation that not only are there clearly new things contained in this book, but also springs have been revealed, and from it there are still many outstanding things."

· On the further development of mathematics (from "The Sublime Higher Mathematics" addressed to Frederick the Great)
"The perceived usefulness of elementary mathematics does not disappear in higher mathematics, but on the contrary, it increases relatively. And mathematics have not yet developed to the stage where very general practical application demands.

· About the joy of solving riddles (from "Letters to a Princess" No. 8)
"If you guess the meaning of the mystery and discover that it is perfectly expressed in the problem of the mystery, you feel the great joy of discovery."

· On the importance of subtle differences (from "Letters to a Princess" No. 118)
"Subtle differences in reports of the same event prove the truth, rather than they weaken it."

· About beings that cannot be reached by arguments (from "Defense against free thinkers")
"When free thinkers try to completely negate the Bible because of contradictions in it (which are obvious to them), they are acting in a most unfair and irresponsible manner. We must admit that they have not been able to solve the problems of geometry, the existence or motion of matter, and that no one has denied the truth or reality of these things. ... There are at least equally great difficulties in the doctrine of revelation, which cannot be reached by argument."

The second volume of "Letters to a Princess" includes many more arguments of Euler. Judging comprehensively from those assertions and the repeated strange errors and descriptions, we deduce that Euler's intentions of the "problems" are as follows.

1. "Introduction" secretly describes the vast world in the subtle mathematics.

2. "Letter" secretly describes the subtle mathematics in the vast world.

3. "Beautiful Relationships" beautifully connects the vast world and the subtle mathematics.

References

[1] Archive staffs “Euler Archive” (http://www.math.dartmouth.edu/~euler/)
[2] W. Dunham, “Introduction to Euler” (Springer-Verlag Tokyo, translated by N. Kurokawa, M. Wakayama, and T. Momotani))
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