

Iwasawa invariants and exceptional pairs of composites of quadratic fields and p -cyclotomic fields

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Abstract

Let p be an odd prime number and ζ_p a primitive p -th root of unity. Following previous computations, we compute the Iwasawa invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$ in the range $|d| < 200$ (resp. $|d| < 10$) and $p < 2,000,000$ (resp. $p < 30,000,000$). We find 15 new exceptional pairs $(p, \chi\omega_p^k)$ including $(p, k, d) = (28,679,999, 2,284,521, -8)$ such that the $\chi\omega_p^{1-k}$ -part of the Iwasawa lambda invariant is two, where χ is the Dirichlet character associated to $\mathbf{Q}(\sqrt{d})$ and ω_p is the Teichmüller character at p . We show differences between exceptional pairs and the other ones in terms of unramified extensions outside p .

Key words: Iwasawa invariant, exceptional pair, unramified extension, ideal class group
2020 Mathematics Subject Classification: Primary 11R23; Secondary 11R18, 11R29, 11R70

1 Introduction

Let χ be the trivial or a primitive quadratic Dirichlet character of conductor $f = f_\chi$, and p an odd prime number such that p does not divide f . Put

*ORCID:0000-0003-3928-9481 The author was partially supported by JSPS KAKENHI Grant Number JP25K06934 and JP26K00600.

$d = d_\chi = \chi(-1)f_\chi$, $K = \mathbf{Q}(\sqrt{d}, \zeta_p)$, and $K_n = \mathbf{Q}(\sqrt{d}, \zeta_{p^{n+1}})$, where ζ_m a primitive m -th root of unity. Let A_n be the p -part of the ideal class group of K_n .

Put $K_\infty = \cup_{n \geq 0} K_n$, $\Gamma = \text{Gal}(K_\infty/K)$, $\Delta = \text{Gal}(K_\infty/\mathbf{Q}_\infty) \simeq \text{Gal}(K_0/\mathbf{Q})$ and $e_\psi = \frac{1}{\#\Delta} \sum_{\delta \in \Delta} \psi(\delta)\delta^{-1}$ for a character ψ of Δ . We put $f_0 = fp$ and identify Δ with a subquotient of $(\mathbf{Z}/f_0\mathbf{Z})^\times$ in the ordinary way.

For a $\mathbf{Z}_p[\Delta]$ -module A , A^ψ denotes $e_\psi A$. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ be the Iwasawa invariants associated to A_n^ψ , i.e.,

$$\#A_n^\psi = p^{\lambda_p(\psi)n + \mu_p(\psi)p^n + \nu_p(\psi)}$$

for sufficiently large n . By Ferrero-Washington's theorem, we have $\mu_p(\psi) = 0$ for all p and ψ .

Assume that ψ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the p -adic L -function is defined as follows. Let $L_p(s, \psi)$ be the p -adic L -function constructed by [9]. By [7, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying

$$G_\psi((1 + f_0)^{1-s} - 1) = L_p(s, \psi)$$

for all $s \in \mathbf{Z}_p$ if $\psi \neq \chi^0$. By [3], p does not divide $G_\psi(T)$. By the p -adic Weierstrass preparation theorem, we can uniquely write $G_\psi(T) = g_\psi(T)u_\psi(T)$, where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Similarly we can define $g_\psi^*(T) \in \mathbf{Z}_p[T]$ from $G_\psi^*(T) \in \mathbf{Z}_p[[T]]$ satisfying

$$G_\psi^*((1 + f_0)^s - 1) = L_p(s, \psi).$$

Put

$$\tilde{\lambda}_p(\psi) = \deg g_\psi(T) = \deg g_\psi^*(T).$$

Put $f_n = f_0 p^n$ and let $\gamma \in \Gamma \simeq \text{Gal}(\cup_{n \geq 0} \mathbf{Q}(\zeta_{f_n})/\mathbf{Q}(\zeta_{f_0}))$ be the generator of Γ such that $\zeta_{f_n}^\gamma = \zeta_{f_n}^{1+f_0}$ for all $n \geq 0$. As usual, we can identify the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the formal power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma = 1 + T$. By this identification, we can consider a $\mathbf{Z}_p[[\Gamma]]$ -module as a Λ -module. For a finitely generated torsion Λ -module A , we define the Iwasawa polynomial $\text{char}_\Lambda(A)$ to be the characteristic polynomial of the action T on $A \otimes \mathbf{Q}_p$ (cf. [18, §13]). Let L_n be the maximal unramified abelian extension of K_n and M_n the maximal abelian extension of K_n unramified outside p . By the class field theory, we have $A_n \simeq \text{Gal}(L_n/K_n)$. Set $L_\infty = \cup_{n \geq 0} L_n$, $M_\infty = \cup_{n \geq 0} M_n$, $X_\infty = \text{Gal}(L_\infty/K_\infty)$ and $Y_\infty = \text{Gal}(M_\infty/K_\infty)$. By the Iwasawa main conjecture proved by [4, 10],

$$\text{char}_\Lambda(X_\infty^{\psi^{-1}\omega}) = g_\psi^*(T) \quad \text{and} \quad \text{char}_\Lambda(Y_\infty^\psi) = g_\psi(T).$$

In the following, we assume that

$$\psi = \chi\omega^k \text{ is even, and } \psi^* = \psi^{-1}\omega = \chi\omega^{p-k} \text{ is odd}$$

with $2 \leq k \leq p-2$. Since p does not divide f ,

$$(C) \quad \psi(p) \neq 1 \text{ and } \psi^*(p) \neq 1.$$

By (C), we have that $A_n^\psi \simeq X_\infty^\psi / \omega_n X_\infty^\psi$ and $A_n^{\psi^*} \simeq X_\infty^{\psi^*} / \omega_n X_\infty^{\psi^*}$, where $\omega_n = (1+T)^{p^n} - 1$ (cf. [6, Lemma 3 and Remark 4]). Moreover, if A_0^ψ is trivial, we have $\lambda(\psi) = \nu(\psi) = 0$, $X_\infty^\psi = \{0\}$, $Y_\infty^\psi \simeq \Lambda/(g_\psi(T))$ and $X_\infty^{\psi^*} \simeq \Lambda/(g_\psi^*(T))$. Put

$$a_0 = a_0(\psi) = L_p(1, \psi) = G_\psi(0) \quad \text{and} \quad b_0 = b_0(\psi) = L_p(0, \psi) = G_\psi^*(0).$$

Note that $v_p(a_0) = v_p(\#\text{Gal}(M_0/K_0)^\psi)$ and $v_p(b_0) = v_p(\#\text{Gal}(L_0/K_0)^{\psi^*})$.

We call $(p, \chi\omega^k)$ exceptional pairs when one of the following conditions holds:

$$[\nu] : \nu(\chi\omega^k) > 0, \quad [a_0] : v_p(a_0) > 1, \quad [b_0] : v_p(b_0) > 1 \quad \text{or} \quad [\text{lmd}] : \tilde{\lambda}(\chi\omega^k) > 1.$$

A lot of number theorists, e.g. Kummer, Vandiver, D.H. Lehmer, E. Lehmer, Selfridge, Nicol, Pollack, Johnson, Wada, Wagstaff, Tanner, Ernvall, Metsänkylä, Buhler, Crandall, Sompolski, Shokrollahi, Hart, Harvey and Ong, computed irregular pairs, and checked that there is no exceptional pair for $d = 1$ and $p < 2^{31} = 2,147,483,648$ (cf. [1, 2, 5]).

In [11, 12, 13, 14, 15], we computed irregular pairs and exceptional pairs for $|d| < 200$ and $p < 1,000,000$, which is useful for computation of the p -part of the ideal class group of K_n and higher K -groups of the integer ring of $\mathbf{Q}(\sqrt{d})$, and for verification of generalized Greenberg's conjecture for $\mathbf{Q}(\sqrt{d}, \zeta_p)$ (cf. [17]).

In this paper, we report new exceptional pairs obtained by further computation in §2, and show differences between exceptional pairs and the other ones in terms of unramified extensions outside p in §3.

2 Numerical results

We use the notation in the previous section. By further computation, we obtain the following.

Proposition 2.1. *For $|d| < 200$ and $1,000,000 < p < 2,000,000$, all exceptional pairs $(p, \chi\omega^k)$ are given in Table 1.*

Table 1: Exceptional pairs for $|d| < 200$ and $1,000,000 < p < 2,000,000$.

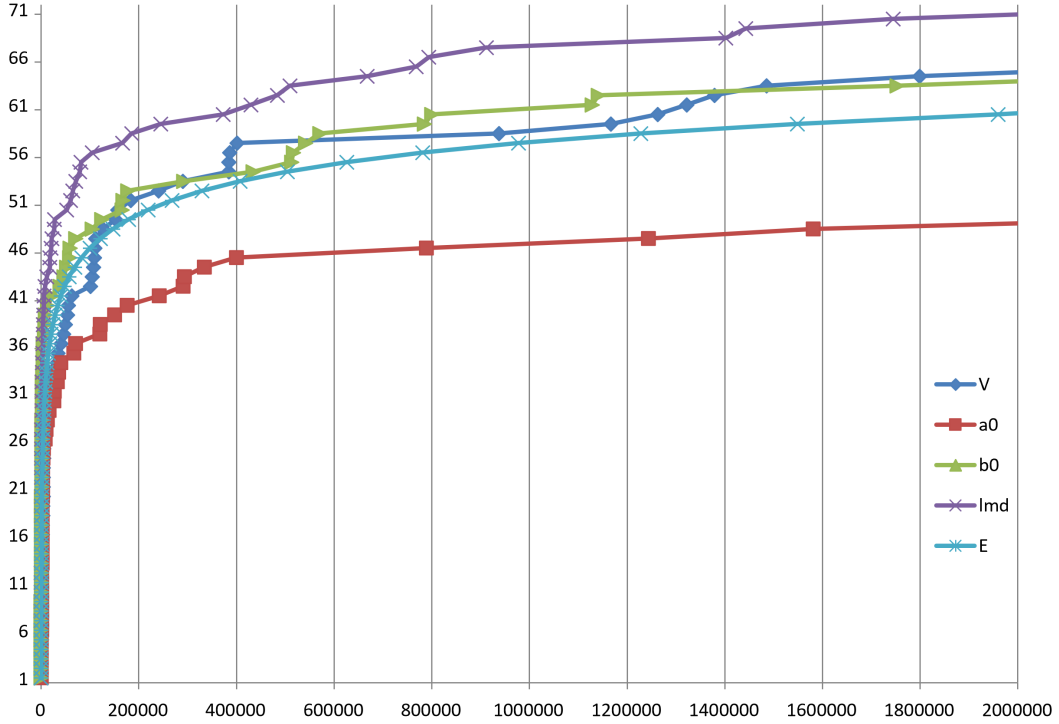
$[\nu]$			$[a_0]$		
p	k	d	p	k	d
1166927	871940	145	1243093	1052719	-183
1262483	816621	-167	1580483	878413	-116
1321927	1087373	-35			
1378691	497941	-167			
1485761	566272	73			
1798813	984380	97			
$[b_0]$			$[lmd]$		
p	k	d	p	k	d
1128233	191315	-67	1402147	944163	-87
1140949	611954	56	1442899	1057203	-151
1752481	557278	24	1744817	928867	-3

In Figure 1, we compare actual numbers of exceptional pairs in the range $f_\chi < 200$ and $200 < p < x$ with the expected number

$$E_1(x) = 123 \sum_{200 < p < x} \frac{p-3}{2} \left(\frac{1}{p}\right)^2,$$

where 123 is the number of the trivial character χ^0 and quadratic characters χ with $f_\chi < 200$.

Figure 1: Actual numbers and the expected number ($200 < p < 2,000,000$).

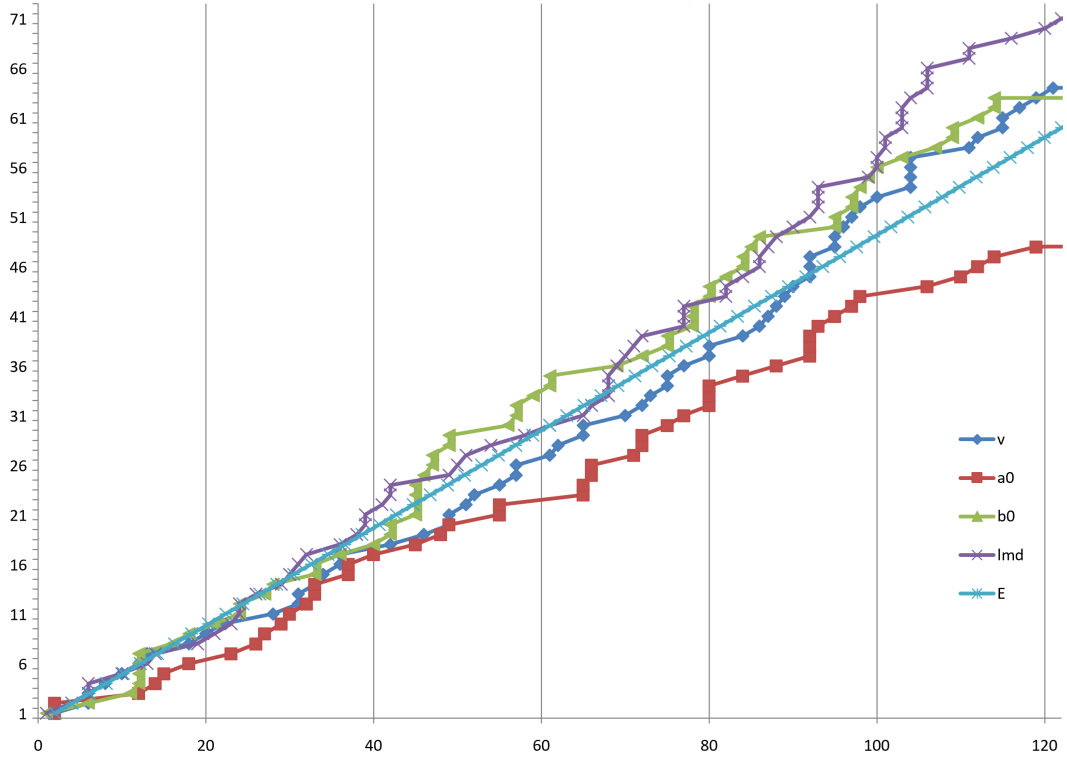


In Figure 2, we compare actual numbers of exceptional pairs in the range $f_\chi < f$ and $200 < p < 2,000,000$ with the expected number

$$E_2(n) = n \sum_{200 < p < 2,000,000} \frac{p-3}{2} \left(\frac{1}{p}\right)^2,$$

where $n = n(f) = \#\{\chi \mid \chi^2 = \chi^0, f_\chi \leq f\}$. From our data, actual numbers still seem to be near the expected number.

Figure 2: Actual numbers and the expected number ($0 \leq n(f) \leq 123$).



In [14, 15], we computed exceptional pairs for $|d| < 10$ and $p < 20,000,000$. The largest prime number among them is 5,911,877 which was found in 2012 and satisfies the condition $[a_0]$ for $(k, d) = (1,629,992, 5)$. By further computation, we obtain the following new exceptional pair which was found in 2024.

Proposition 2.2. *For $|d| < 10$ and $6,000,000 < p < 30,000,000$, there is only one exceptional pair $(p, \chi\omega^k)$ which is given in Table 2.*

Table 2: The exceptional pair for $|d| < 10$ and $6,000,000 < p < 30,000,000$.

[lmd]		
p	k	d
28679999	2284521	-8

For details of computation, see [15]. The programs and further data have been available in our web page: <https://hiro2.pm.tokushima-u.ac.jp/~hiroki/major/galois1-e.html>. These data were obtained by seven personal computers with 76 cores (152 threads) for about five years. We computed twice for $|d| < 200$ and $p < 2,000,000$, and only once for $|d| < 10$ and $p < 30,000,000$.

Though temporary hardware errors are rare, they could occur in long-term computation. We are afraid that they occur in the latter computation. However, these rare errors would not affect Proposition 2.2, because the expected number is very small. In fact,

$$\sum_{x_0 < p < x_1} \frac{p-3}{2} \left(\frac{1}{p}\right)^2 \approx \frac{1}{2}(\log \log 30,000,000 - \log \log 6,000,000) = 0.04907 \dots$$

for $x_0 = 6,000,000$ and $x_1 = 30,000,000$. It might be hard to find a new exceptional pair for $|d| < 10$, because

$$\sum_{x_0 < p < x_1} \frac{p-3}{2} \left(\frac{1}{p}\right)^2 \approx \frac{1}{2}(\log \log 100,000,000 - \log \log 30,000,000) = 0.03379 \dots$$

for $x_0 = 30,000,000$ and $x_1 = 100,000,000$.

3 Unramified abelian extensions outside p

We use the notation in the previous sections and assume that $\psi = \chi\omega^k$ satisfies (C). Then, if $(p, \chi\omega^k)$ is a regular pair, we have $v_p(a_0(\psi)) = 0$, $v_p(b_0(\psi)) = 0$ and $\tilde{\lambda}(\psi) = \lambda(\psi^*) = \nu(\psi) = \nu(\psi^*) = 0$, where $\psi^* = \psi^{-1}\omega = \chi\omega^{1-k}$. Therefore,

Case [R]

$$\begin{aligned} X_\infty^\psi &= \{0\}, & Y_\infty^\psi &= \{0\}, \\ X_\infty^{\psi^*} &= \{0\} & \text{and} & Y_\infty^{\psi^*} \simeq \Lambda. \end{aligned}$$

In the following, we assume that $(p, \chi\omega^k)$ is an irregular pair. Then, $v_p(a_0(\psi)) > 0$, $v_p(b_0(\psi)) > 0$ and $\tilde{\lambda}(\psi) = \lambda(\psi^*) > 0$, where $\psi^* = \chi\omega^{1-k}$. Let D_n be the subgroup of A_n which is generated by the class of prime ideals lying above p and put $A'_n = A_n/D_n$. Let E_n be the unit group of K_n and E'_n the p -unit group of K_n . By (C), $D_n^\psi = D_n^{\psi^*} = \{0\}$, $A_n^\psi \simeq A_n'^\psi$, $A_n^{\psi^*} \simeq A_n'^{\psi^*}$ and $H^1(K_n/K, E_n)^\psi \simeq H^1(K_n/K, E'_n)^\psi$.

First, we consider the case that (p, ψ) is not an exceptional pair. By [8, Theorem 12 and Lemma 12], we have the following Λ -module structures of Galois groups (cf. case (A) in [6]).

Case [nE]

$$\begin{aligned} X_\infty^\psi &= \{0\}, & Y_\infty^\psi &\simeq \Lambda/(T - \alpha), \\ X_\infty^{\psi^*} &\simeq \Lambda/(T - \alpha^*) & \text{and} & \quad Y_\infty^{\psi^*} \simeq \Lambda \end{aligned}$$

for some $\alpha, \alpha^* \in p\mathbf{Z}_p \setminus p^2\mathbf{Z}_p$.

Next, we consider four cases that (p, ψ) satisfies only one of conditions $[\nu]$, $[a_0]$, $[b_0]$ and $[\text{lmd}]$.

Case $[\nu]$

(i) If Greenberg's conjecture is true for p and ψ ,

$$\begin{aligned} X_\infty^\psi &\simeq \Lambda/(T - \alpha, p^\nu), & Y_\infty^\psi &\simeq \Lambda/(T - \alpha), \\ X_\infty^{\psi^*} &\simeq \Lambda/(T - \alpha^*) & \text{and} & \quad Y_\infty^{\psi^*} \simeq (T, p) \subset \Lambda \end{aligned}$$

for some $\alpha, \alpha^* \in p\mathbf{Z}_p \setminus p^2\mathbf{Z}_p$ and $\nu \in \mathbf{Z}_{\geq 1}$.

(ii) If Greenberg's conjecture is false for p and ψ ,

$$\begin{aligned} X_\infty^\psi &\simeq \Lambda/(T - \alpha), & Y_\infty^\psi &\simeq \Lambda/(T - \alpha), \\ X_\infty^{\psi^*} &\simeq \Lambda/(T - \alpha^*) & \text{and} & \quad Y_\infty^{\psi^*} \simeq \Lambda/(T - \alpha^*) \oplus \Lambda \end{aligned}$$

for some $\alpha, \alpha^* \in p\mathbf{Z}_p \setminus p^2\mathbf{Z}_p$.

Case $[a_0]$

$$\begin{aligned} X_\infty^\psi &= \{0\}, & Y_\infty^\psi &\simeq \Lambda/(T - \alpha), \\ X_\infty^{\psi^*} &\simeq \Lambda/(T - \alpha^*) & \text{and} & \quad Y_\infty^{\psi^*} \simeq \Lambda \end{aligned}$$

for some $\alpha \in p^2\mathbf{Z}_p$, $\alpha^* \in p\mathbf{Z}_p \setminus p^2\mathbf{Z}_p$.

Case $[b_0]$

$$\begin{aligned} X_\infty^\psi &= \{0\}, & Y_\infty^\psi &\simeq \Lambda/(T - \alpha), \\ X_\infty^{\psi^*} &\simeq \Lambda/(T - \alpha^*) & \text{and} & \quad Y_\infty^{\psi^*} \simeq \Lambda \end{aligned}$$

for some $\alpha \in p\mathbf{Z}_p \setminus p^2\mathbf{Z}_p$, $\alpha^* \in p^2\mathbf{Z}_p$.

Case $[\text{lmd}]$

$$\begin{aligned} X_\infty^\psi &= \{0\}, & Y_\infty^\psi &\simeq \Lambda/(g(T)), \\ X_\infty^{\psi^*} &\simeq \Lambda/(g^*(T)) & \text{and} & \quad Y_\infty^{\psi^*} \simeq \Lambda \end{aligned}$$

for some distinguished polynomials $g(T), g^*(T) \in \mathbf{Z}_p[T]$ such that $g(0), g^*(0) \in p\mathbf{Z}_p \setminus p^2\mathbf{Z}_p$, $\deg g(T) = \deg g^*(T) \geq 2$.

From the above Λ -structures, we immediately obtain the following.

Theorem 3.1. *Assume that ψ satisfies (C) and that (p, ψ) is an irregular pair. In Case $[\nu]$, there is an unramified abelian p -extension L'_n of K_n such that $\text{Gal}(L'_n/K_n)^\psi$ is not trivial for $n \geq 0$. In Case $[a_0]$, there is an unramified abelian p -extension M'_n of K_n outside p such that $|\text{Gal}(M'_n/K_n)^\psi| \geq p^{n+2}$ for $n \geq 0$. In Case $[b_0]$, there is an unramified abelian p -extension L'_n of K_n such that $|\text{Gal}(L'_n/K_n)^{\psi^*}| \geq p^{n+2}$ for $n \geq 0$. In Case $[\text{lmd}]$, there is an unramified abelian p -extension L'_n of K_n such that $|\text{Gal}(L'_n/K_n)^{\psi^*}| \geq p^{2n+1}$ for $n \geq 1$. On the other hand, those extensions do not exist if (p, ψ) is not an exceptional pair.*

In the above theorem, we use the Δ -decomposition of Galois groups. Let us give another statement which characterizes Case $[\text{lmd}]$. We consider the following two typical cases with $X_\infty \simeq \mathbf{Z}_p^\lambda$.

Case $[\text{lmd} \times 1]$

$$X_\infty^{\psi^*} \simeq \Lambda/(g^*(T)) \quad (\text{Case } [\text{lmd}]) \quad \text{and} \quad X_\infty = X_\infty^{\omega^{1-k}}.$$

Case $[\text{nE} \times \lambda]$

$$X_\infty^{\chi\omega^{1-k_i}} \simeq \Lambda/(T - \alpha_i^*) \quad (\text{Case } [\text{nE}]) \quad \text{and} \quad X_\infty = \bigoplus_{i=1}^{\lambda} X_\infty^{\omega^{1-k_i}}.$$

Theorem 3.2. *In Case $[\text{lmd} \times 1]$, there is no unramified abelian p -extension L'_n of K_n such that*

$$\text{Gal}(L'_n/K_n) \simeq (\mathbf{Z}/p^{a_1}\mathbf{Z}) \oplus (\mathbf{Z}/p^{a_2}\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/p^{a_r}\mathbf{Z})$$

with $r \leq \lambda$ and $a_1 \leq a_2 \leq \cdots \leq a_r$ such that $a_r - a_1 \geq 2$ and that L'_n is a Galois extension of K_0 for $n \geq 0$. On the other hand, in Case $[\text{nE} \times \lambda]$, there exists such an unramified abelian extension of K_n which is a Galois extension of K_0 for $n \geq 1$.

Proof. Put $X' = \text{Gal}(L_\infty/K_\infty)^{\psi^*}$. Then, the subgroup of X' which corresponds to the maximal abelian extension of K_n is $\omega_n X'$. Further, subgroups of X' which correspond to Galois extensions of K_0 are Λ -submodules of X' . We show that a Λ -submodule M' of $M = \Lambda/(g^*(T))$ is determined by its index when $g^*(T)$ is an Eisenstein polynomial, along the proof in [16, Theorem 2.4]. Put $\lambda = \deg g^*(T) \geq 2$. Let $h(T) \in \Lambda \setminus (g^*(T))$. By p -adic Weierstrass preparation theorem, $h(T) \equiv p^{s'-1}u(T)f(T) \pmod{(g^*(T))}$, where $s' \geq 1$, $u(T) \in \Lambda^\times$, and $f(T)$ is a distinguished polynomial of degree $b' < \lambda$. Then, we have

$$\begin{aligned} (g^*(T), h(T)) &= (g^*(T), p^{s'-1}f(T)) \\ &= (g^*(T), p^{s'-1}f(T), p^{s'-1}g^*(T) - p^{s'-1}T^{\lambda-b'}f(T)) \\ &= (g^*(T), p^{s'-1}T^{b'}, p^{s'}). \end{aligned}$$

Let s be the minimum number of such s' contained in M' modulo $(g^*(T))$. Further let b be the minimum number of such b' with $s' = s$ contained in M' modulo $(g^*(T))$. M' is determined by s and b , i.e., $|M/M'| = p^{\lambda(s-1)+b}$. Then, as an abelian group,

$$M/M' \simeq (\mathbf{Z}/p^{(s-1)}\mathbf{Z})^{\lambda-b} \oplus (\mathbf{Z}/p^s\mathbf{Z})^b.$$

This implies the non-existence of a Galois extension of K_0 which satisfies the condition in the theorem for $n \geq 0$.

On the other hand, for $M = \bigoplus_{i=1}^{\lambda} \Lambda/(T - \alpha_i^*)$, $\langle p^{a_1}e_1, p^{a_2}e_2, \dots, p^{a_{\lambda}}e_{\lambda} \rangle$ is a Λ -submodule of M for any $a_i \in \mathbf{Z}_{\geq 0}$, where $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_{\lambda} = (0, 0, \dots, 0, 1)$. Since $\omega_1 M = p^2 M$, this implies the existence of a Galois extension of K_0 which satisfies the condition in the theorem for $n \geq 1$. $\square$

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